Uniqueness of a solution in 11.9

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Wednesday, 23 September 2015

One of the homework problems was to show that $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ is exactly the function $g(x) = e^x$ by showing that they are both solutions to the differential equation $y' = y$. While it is straightforward to see that $f'(x) = f(x)$ and $g'(x) = g(x)$, what isn't so straightforward is to see $f(x) = g(x)$.

Since $f(0) = 1$ and $g(0) = 1$, we could use the uniqueness theorem for differential equations to conclude $f(x) = g(x)$. However, as far as I can tell, this theorem is only mentioned by name in a side note in a margin of chapter 9.

An alternative method is to try solving $y' = y$ directly: This is the equation $\frac{dy}{dx} = y$, and it is a separable equation, which was possibly discussed in Math 1A. We can solve this by rewriting it in terms of differentials as $\frac{1}{y}\,dy = dx$, then integrating both sides:

$$\int \frac{1}{y}\,dy = \int dx$$

which gives $\ln y = x + C$, implying $y = e^{x+C}$. Since $f(0) = 1$, then $C = 0$ gives the equality $f(x) = e^x$. (This relies on you believing that differential equations solved that way cannot be solved in any other way.) Stewart probably intended something like this.

However, we could do better. A colleague showed me a trick from the field of partial differential equations. If we want to show that $f(x) = e^x$, let us try to show that $\frac{f(x)}{e^x}$ is a constant. We compute

$$\begin{aligned} \frac{d}{dx} e^{-x} f(x) &= e^{-x} f'(x) - e^{-x} f(x) \\ &= e^{-x} f(x) - e^{-x} f(x) \\ &= 0 \end{aligned}$$

Since its derivative is zero, by the mean value theorem we see that $e^{-x} f(x) = C$ for some C (in other words, integrate both sides). Thus, $f(x) = Ce^x$. But, since $f(0) = 1$, it must be that $C = 1$, hence $f(x) = e^x$.

Notice that all we are relying on about $f(x)$ is that it is its own derivative. We use the series definition to verify this fact, and then never touch the definition of f again. A principle: to show two things are the same, find a key property they share, then demonstrate that there can only be one thing which satisfies that property.

One last point we really ought to check: that the domain of f is all of $\mathbb{R}$, otherwise it is not actually the exponential function. If we do the ratio test, we get the ratio $|\frac{x}{n}|$, which tends to 0 as $n \to \infty$ no matter the value of x, hence the radius of convergence is ∞.