A series mentioned in lecture (update)

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The following series was mentioned in lecture:

$$\sum_{n=1}^{\infty} \frac{1-2^{(-1)^n}}{n}$$

In a previous note, we found this diverges by an argument which appeals to partial sums. While it is a good kind af argument to understand, there is a simpler way to come to the same conclusion.

The general technique is that if we add or subtract a convergent series from another series, and the result is divergent, then it must be that the second series is divergent as well. We know that $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is a convergent series. Observe that $1 - 2^{(-1)^n}$ is the sequence $-\frac{1}{2}, \frac{2}{2}, -\frac{1}{2}, \frac{2}{2}, \dots$. This means if we subtract the sequence $\frac{(-1)^n}{2}$ from this, every other term is zero: $0, \frac{1}{2}, 0, \frac{1}{2}, \dots$. So, one can see that

$$\begin{aligned}\left(\sum_{n=1}^{\infty} \frac{1-2^{(-1)^n}}{n}\right) - \left(\frac{1}{2}\sum_{n=1}^{\infty} \frac{(-1)^n}{n}\right) &= 0 - \frac{1}{2\cdot 2} + 0 - \frac{1}{2\cdot 4} + 0 - \frac{1}{2\cdot 6} + \cdots \\ &= -\frac{1}{2\cdot 2} - \frac{1}{2\cdot 4} - \frac{1}{2\cdot 6} - \cdots \\ &= -\frac{1}{4}\sum_{n=1}^{\infty} \frac{1}{n}\end{aligned}$$

which of course diverges.

A stickler for details would verify that dropping zeros out of a series does not change the convergence or divergence. This fact is relatively easy to verify, the hardest part being how to explain it succinctly, so if you are not such a stickler, it is a safe fact to make use of.

In essence, given a converging series $\sum a_n$, make a new series $\sum b_n$ where b_n skips some set of a_n which are zero. For each $\varepsilon > 0$, there is some point N where the partial sums $\sum a_n$ eventually get within ε of the sum of the series, and this N will work for $\sum b_n$ as well because each term of $\{b_n\}$ comes no later than its corresponding term in $\{a_n\}$. This shows it even has the same value (as one would expect). To finish this up, you would have to say a similar thing for convergence of $\sum b_n$ meaning convergence of $\sum a_n$.

Alternatively, the proof for $\sum_{n=1}^{\infty} \frac{1}{n}$ could be adapted to this specific instance, but that is more work than necessary (unless you simply use the incantation "and this series diverges by a similar argument as that for the harmonic series"; as long as you say something like that in the right place, it demonstrates you know what you're talking about).