A series mentioned in lecture

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The following series was mentioned in lecture:

$$\sum_{n=1}^{\infty} \frac{1 - 2^{(-1)^n}}{n}$$

Does it converge or not? The professor indicated determining (and proving) was somewhat difficult. We will see it diverges.

First, for some clarity, the quantity $2^{(-1)^n}$ is equal to $2^{((-1)^n)}$, since exponentiation groups according to the size of the text (so, from right to left). These give the sequence $\frac{1}{2}, 2, \frac{1}{2}, 2, \dots$.

Hence, the series is

$$\frac{1}{2 \cdot 1} - \frac{1}{2} + \frac{1}{2 \cdot 3} - \frac{1}{4} + \dots$$

Let s_n denote the partial sums. We will show that $s_{2n} \to -\infty$ as $n \to \infty$. For convenience, we will define $h_n = \sum_{i=1}^{n} \frac{1}{i}$, as usual. First, notice

$$\begin{aligned}
s_{2n} &= \frac{1}{2 \cdot 1} - \frac{1}{2} + \frac{1}{2 \cdot 3} - \frac{1}{4} + \dots + \frac{1}{2 \cdot (2n-1)} - \frac{1}{2n} \\
&= \left(\frac{1}{2 \cdot 1} + \frac{1}{2 \cdot 3} + \dots + \frac{1}{2 \cdot (2n-1)} \right) - \left(\frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2n} \right) \\
&= \frac{1}{2} \left(\frac{1}{1} + \frac{1}{3} + \dots + \frac{1}{2n-1} \right) - \frac{1}{2} \left(\frac{1}{1} + \frac{1}{2} + \dots + \frac{1}{n} \right).
\end{aligned}$$

The first sum is the odd terms of the partial sum of a harmonic series, which can be obtained by subtracting half of a harmonic series (so it is $h_{2n} - \frac{1}{2} h_n$). The second sum is just from a harmonic series. So,

$$\begin{aligned}
s_{2n} &= \frac{1}{2} \left(h_{2n} - \frac{1}{2} h_n \right) - \frac{1}{2} h_n \\
&= \frac{1}{2} h_{2n} - \frac{3}{4} h_n.
\end{aligned}$$

Previously we have shown that $\ln n \leq h_n \leq \ln(n+1)$ by comparing with integrals, so

$$\begin{aligned}
s_{2n} = \frac{1}{2} h_{2n} - \frac{3}{4} h_n &\leq \frac{1}{2} \ln(2n+1) - \frac{3}{4} \ln n \\
&= \ln \frac{(2n+1)^{1/2}}{n^{3/4}},
\end{aligned}$$

which, as $n \to \infty$, gives $s_{2n} \to -\infty$. The series diverges.

Do not be afraid to look at partial sums. In a previous discussion section meeting, the following series was considered:

$$\sum_{n=1}^{\infty} a_n \text{ where } a_n = \begin{cases} \frac{1}{n} & n \text{ odd} \\ -\frac{1}{n^2} & n \text{ even} \end{cases}$$

In this case, we can use the fact that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (let its value be c), and so the sum of the even terms is less than c. Hence, $s_{2n} > h_{2n} - \frac{1}{2}h_n - c > \ln(2n) - \frac{1}{2}\ln(n+1) - c = \ln \frac{2n}{(n+1)^{1/2}} - c$. As $n \to \infty$, the right-hand side approaches ∞, so $s_{2n} \to \infty$ as well. We take care here to use partial sums because, in general, series are not robust to rearrangements.