Midterm 2 Solutions

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Here is the full midterm:

Problems

Midterm 2, Math1B, 20151026 11-12

$\boxed{\text{Correct answers without explanation = no credit}}$

1. Show that $0 < \tan(x) - x < \frac{8}{9}x^3$ when $0 < x < \frac{\pi}{6}$.
2. Show that $\int_0^{a/2} \sqrt{2ax - x^2}\, dx = a^2 \left[\frac{\pi}{6} - \frac{\sqrt{3}}{8}\right]$ when $a > 0$.
3. Assume that the time to solve *one* midterm problem is exponentially distributed with mean $= 10$ minutes. The probability distribution of the time for solving *two* midterm problems is then given by Erlang's probability density function

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ \lambda^2 x e^{-\lambda x} & \text{if } x \geq 0 \text{ and } \lambda = \frac{1}{10}\,\text{min}^{-1}. \end{cases}$$

   (a) Show that $\int_0^\infty x^k e^{-\lambda x}\, dx = \frac{k!}{\lambda^{k+1}}$ for $k = 1, 2, \ldots$ and $\lambda > 0$.

   (b) Show that the mean and the standard deviation of the time to solve *two* midterm problems are $\mu = 20\,\text{min}$ and $\sigma = 10 \cdot \sqrt{2}\,\text{min}$.

   Remember: $\mu = \int_{-\infty}^{\infty} x f(x)\, dx$ and $\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx$.

Solutions

Like in the midterm 2 review, I am experimenting with giving solutions with all steps enumerated. This is in more detail than is expected on an exam, but the intended audience for this is a student curious about *all* the steps. Of course, these are just one set of surely many ways to solve the problems.

1. (a) The only theorems involving inequalities are comparison tests or Taylor's inequality. The cube on the right side paired with the fact that the polynomial x is subtracted from $\tan x$ suggests that it might be Taylor's inequality, at least for the right-hand inequality.

   (b) Let us compute the $n = 2$ Taylor polynomial:

      i. $\frac{d}{dx} \tan x = \sec^2 x$
      ii. $\frac{d^2}{dx^2} \tan x = 2 \sec^2 x \tan x$
      iii. Evaluating the function, its derivative, and its second derivative at 0 gives the Taylor polynomial $T_2(x) = 0 + \frac{1}{1!}x + \frac{0}{2!}x^2 = x$.

   (c) Hence, $R_2(x) = \tan(x) - x$.

   (d) Claim: the third derivative is increasing on the domain $0 \leq x \leq \frac{\pi}{6}$.

      i. $\frac{d^3}{dx^3} \tan x = 2(2 \sec^2 \tan^2 x + \sec^4 x)$
      ii. This can be rewritten using $\sec^2 x + 1 = \tan^2 x$ as $\frac{d^3}{dx^3} \tan x = 2(1 + 4 \tan^2 x + 3 \tan^4 x)$.
      iii. The function $\tan x$ is increasing on the domain $0 \leq x \leq \frac{\pi}{6}$ since its derivative is $\sec^2 x = \frac{1}{\cos^2 x}$, and on $0 \leq x \leq \frac{\pi}{6}$, $\cos x$ is positive.
      iv. The square of an increasing function is still increasing, and so is the square of a square. The sum of increasing functions is an increasing function, and so is adding a constant to an increasing function.
      v. Therefore $2(1 + 4 \tan^2 x + 3 \tan^4 x)$ is increasing on the domain $0 \leq x \leq \frac{\pi}{6}$.

   (e) The third derivative is an even function: $2(1 + 4 \tan^2(-x) + 3 \tan^4(-x)) = 2(1 + 4(-\tan x)^2 + 3(-\tan x)^4) = 2(1 + 4 \tan^2 x + 3 \tan^4 x)$.

   (f) For each $0 < d < \frac{\pi}{6}$.

      i. Since the third derivative is increasing, for all x satisfying $0 \leq x \leq d$ we have $2(1 + 4 \tan^2 x + 3 \tan^4 x) \leq 2(1 + 4 \tan^2 d + 3 \tan^4 d)$.
      ii. Again since it is increasing, since $d < \frac{\pi}{6}$, $2(1 + 4 \tan^2 d + 3 \tan^4 d) < 2(1 + 4 \tan^2 \frac{\pi}{6} + 3 \tan^4 \frac{\pi}{6}) = 2(1 + 4(\frac{1}{\sqrt{3}})^2 + 3(\frac{1}{\sqrt{3}})^4) = \frac{48}{9}$.
      iii. By evenness, $2(1+4\tan^2 d+3\tan^4 d)$ is an upper bound for the third derivative on the domain $-d \leq x \leq d$. And this upper bound is strictly less than $\frac{48}{9}$.

   (g) For all x satisfying $0 < x < \frac{\pi}{6}$:

      i. There is some d between x and $\frac{\pi}{6}$ (there are infinitely many; just choose one). This number satisfies $0 < d < \frac{\pi}{6}$ since $0 < x$.

ii. Let $M = 2(1 + 4\tan^2 d + 3\tan^4 d)$. We have shown that this is an upper bound for the third derivative. Since the third derivative is always positive, this is also an upper bound for the absolute value of the third derivative.

iii. Therefore, by Taylor's inequality, $|R_2(x)| \leq \frac{M}{3!}|x|^3$, which is satisfied by the current x since $-d \leq x \leq d$.

iv. In fact, $M < \frac{48}{9}$ and x is positive, so $|R_2(x)| < \frac{48/9}{3!}x^3 = \frac{8}{9}x^3$.

v. And since absolute value can only make something larger, $R_2(x) < \frac{8}{9}x^3$.

(h) Since we showed this was true whenever $0 < x < \frac{\pi}{6}$, we can say that $R_2(x) < \frac{8}{9}x^3$ for all $0 < x < \frac{\pi}{6}$, and because $R_2(x) = \tan(x) - x$, this proves the right-hand inequality.

(i) For the left-hand inequality, notice that the derivative of $\tan(x)-x$ is $\sec^2(x)-1 = \tan^2 x$.

(j) Since $\tan^2 x > 0$ for all $0 < x < \frac{\pi}{6}$, $\tan(x) - x$ is increasing.

(k) Because $\tan(0) - 0 = 0$, we see that $\tan(x) - x > 0$ for all x satisfying $0 < x < \frac{\pi}{6}$.

(l) Putting the two parts of the inequality together, we conclude $0 < \tan(x)-x < \frac{8}{9}x^3$ for all x satisfying $0 < x < \frac{\pi}{6}$.

2. (a) This is a square root of a quadratic, so it is likely a trigonometric substitution.

(b) Trigonometric substitutions in the book were all involved integrands of the form $\sqrt{\pm c^2 \pm x^2}$, so we will complete the square.

i. Just for the sake of reducing it to a previous type of problem, we will complete the square of $x^2 - 2ax$ then take the negative of that.

ii. The linear coefficient is $-2a$, and half that is $-a$, so the square part is of the form $(x-a)^2 = x^2 - 2ax + a^2$. Hence, $x^2 - 2ax = (x-a)^2 - a^2$.

iii. Therefore $2ax - x^2 = a^2 - (x-a)^2$.

(c) Thus, the integral we are to compute is $\int_0^{a/2} \sqrt{a^2 - (x-a)^2}\, dx$.

(d) Let $x - a = a\sin\theta$. Then $dx = a\cos\theta\, d\theta$.

(e) When $x = 0$, $\theta = -\frac{\pi}{2}$ for $0 - a = a\sin(-\frac{\pi}{2})$. When $x = \frac{a}{2}$, $\theta = -\frac{\pi}{6}$ for $\frac{a}{2} - a = a\sin(-\frac{\pi}{6})$.

(f) The substitution gives the integral $\int_{-\pi/2}^{-\pi/6} a^2\cos^2\theta\, d\theta$.

(g) Recall the that $\cos^2\theta = \frac{1}{2}(1 + \cos(2\theta))$.

(h) Then this is $a^2\int_{-\pi/2}^{-\pi/6} \frac{1}{2}(1+\cos(2\theta))\, d\theta = a^2\left[\frac{1}{2}(\theta + \frac{1}{2}\sin(2\theta))\right]_{-\pi/2}^{-\pi/6}$

(i) Evaluating, this is $a^2(\frac{1}{2}(\frac{-\pi}{6} + \frac{1}{2}\frac{-\sqrt{3}}{2}) - \frac{1}{2}(\frac{-\pi}{2} + \frac{1}{2}\cdot 0)) = a^2(\frac{\pi}{6} - \frac{\sqrt{3}}{8})$.

3. (a) We are considering $\int_0^\infty x^k e^{-\lambda x}\, dx$ first, for $k = 1, 2, \ldots$ and $\lambda > 0$.

i. Since this is improper, let b be the upper bound for a new definite integral, then we will take the limit $b \to \infty$. This is so that we are careful when doing integration by parts.

ii. Let $u = x^k$ and $dv = e^{-\lambda x}\,dx$. With these, $du = kx^{k-1}\,dx$ and $v = -\frac{1}{\lambda}e^{-\lambda x}$. Hence, $\int_0^b x^k e^{-\lambda x}\,dx = \left[-\frac{1}{\lambda}x^k e^{-\lambda x}\right]_0^b + \frac{k}{\lambda}\int_0^b x^{k-1}e^{-\lambda x}\,dx$.

iii. We see $\left[-\frac{1}{\lambda}x^k e^{-\lambda x}\right]_0^b = -\frac{1}{\lambda}b^k e^{-\lambda b}$, and $\lim_{b\to\infty}\frac{b^k}{e^{\lambda b}} = 0$. I'm pretty sure this is something you've shown before, but one way to prove it is the following:

A. The statement we would like to prove is that for all $k = 0, 1, \ldots,$ then $\lim_{b\to\infty}\frac{b^k}{e^{\lambda b}} = 0$.

B. For $k = 0$, this limit is $\lim_{b\to\infty}\frac{1}{e^{\lambda b}}$, which is zero because $e^{\lambda b}$ increases without bound. (You could give a proof of this limit. A schmancy way to prove this is to note that $\sum_{n=1}^{\infty}\frac{1}{e^{\lambda n}}$ converges because $e^{\lambda} > 0$, so $\lim_{n\to\infty}\frac{1}{e^{\lambda n}} = 0$, and since $f(x) = \frac{1}{e^{\lambda x}}$ is continuous, $\lim_{b\to\infty}\frac{1}{e^{\lambda b}} = 0$ as well.)

C. For $k > 0$, if the limit for $k-1$ is zero: This limit is $\lim_{b\to\infty}\frac{b^k}{e^{\lambda b}}$, which is an $\frac{\infty}{\infty}$ indeterminate form, so by l'Hopital's rule, this limit is $\frac{k}{\lambda}\lim_{b\to\infty}\frac{b^{k-1}}{e^{\lambda b}} = \frac{k}{\lambda}\cdot 0 = 0$.

D. By induction, the limit is zero for all $k = 0, 1, \ldots$.

Alternatively, you can use the ratio test:

A. $\left|\frac{(n+1)^k/e^{\lambda(n+1)}}{n^k/e^{\lambda n}}\right| = \frac{(n+1)^k}{n^k e^{\lambda}}$, which approaches $\frac{1}{e^{\lambda}}$ as $n \to \infty$.

B. Since $e^{\lambda} > 1$, the ratio test says $\sum_{n\to\infty}\frac{n^k}{e^{\lambda n}}$ converges.

C. Hence $\lim_{n\to\infty}\frac{n^k}{e^{\lambda n}} = 0$.

D. Since $f(x) = \frac{1}{e^{\lambda x}}$ is continuous, $\lim_{b\to\infty}\frac{b^k}{e^{\lambda b}} = 0$, too. (The first was limit where n is an integer, and the second is limit where b is any real number.)

iv. Thus, $\int_0^\infty x^k e^{-\lambda x}\,dx = \frac{k}{\lambda}\int_0^\infty x^{k-1}e^{-\lambda x}\,dx$.

v. Thus, $\int_0^\infty x^k e^{-\lambda x}\,dx = \frac{k}{\lambda}\cdot\frac{k-1}{\lambda}\cdot\frac{k-2}{\lambda}\cdots\frac{2}{\lambda}\cdot\frac{1}{\lambda}\cdot\int_0^\infty e^{-\lambda x}\,dx = \frac{k!}{\lambda^k}\int_0^\infty e^{-\lambda x}\,dx$.

vi. $\int_0^\infty e^{-\lambda x}\,dx = \lim_{b\to\infty} -\frac{1}{\lambda}(e^{-\lambda b} - 1) = -\frac{1}{\lambda}$.

vii. Therefore $\int_0^\infty x^k e^{-\lambda x}\,dx = \frac{k!}{\lambda^{k+1}}$.

(b) i. The probability distribution for solving two midterm problems is given to us (Erlang's probability density function).

ii. The mean first:

A. We plug the given f into the given formula $\mu = \int_{-\infty}^{\infty} x f(x)\,dx$.

B. Since $f(x)$ is 0 for all $x < 0$, $\mu = \int_0^\infty x(\lambda^2 x e^{-\lambda x})\,dx$.

C. $\lambda = \frac{1}{10}\,\text{min}^{-1}$ is given to us.

D. This integral, simplified, is $\lambda^2\int_0^\infty x^2 e^{-\lambda x}\,dx$, so we may use (a) to compute the integral, with $k = 2$.

E. So it is $\lambda^2\frac{2!}{\lambda^3} = \frac{2}{\lambda} = 20\,\text{min}$.

iii. The standard deviation second:

A. We again plug the given f into the given formula $\sigma^2 = \int_{-\infty}^{\infty}(x-\mu)^2 f(x)\,dx$.

B. This is $\int_0^\infty (x-20)^2(\lambda^2 x e^{-\lambda x})\,dx$.

C. Expanding, this is $\lambda^2\left(\int_0^\infty x^3 e^{-\lambda x}\,dx - 40\int_0^\infty x^2 e^{-\lambda x}\,dx + 400\int_0^\infty x e^{-\lambda x}\,dx\right)$.

D. And using the formula from (a) with $k = 3, 2, 1$, this is $\lambda^2(\frac{3!}{\lambda^4} - 40\frac{2!}{\lambda^3} + 400\frac{1!}{\lambda^2}) = \frac{6}{\lambda^2} - \frac{80}{\lambda} + 400 = 600 - 800 + 400 = 200\,\mathrm{min}^2$.

E. This is σ^2. Therefore $\sigma = 10\sqrt{2}\,\mathrm{min}$.