Midterm 2 Review

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1 Topics

This is a short overview of topics we have covered since the last midterm. Of course, older topics (sequences, series, convergence) are still relevant to the work we have been doing, so it would be good to review them.

1.1 Techniques of Integration (Chapter 7)

1. Page 463, which is the first page of Chapter 7, has a list of useful antiderivatives. They seem to be listed on 495, too.

2. Again, remember integration by parts ($\int u\,dv = uv - \int v\,du$). Be familiar with how to solve for an antiderivative for when the integration enters a loop. (What is an example of this?)

3. Remember symmetry. For instance, $\int_{-a}^{a} x\sqrt{1-x^2}\,dx = 0$.

4. Remember that definite integrals can be found by geometric reasoning. For instance, $\int_0^a \sqrt{1-x^2}\,dx = \frac{1}{2}\sin^{-1}(\frac{1}{a}) + \frac{1}{2}a\sqrt{1-a^2}$ by breaking the region of integration into a triangle and a sector of a circle.

5. Trigonometric integrals use trigonometric identites, such as $\sin^2\theta + \cos^2\theta = 1$, $\tan^2\theta + 1 = \sec^2\theta$, $\sin^2\theta = \frac{1}{2}(1-\cos(2\theta))$, $\cos^2\theta = \frac{1}{2}(1+\cos(2\theta))$, and $\sin\theta\cos\theta = \frac{1}{2}\sin(2\theta)$ (strategies are listed on 473 and 474). There are also identities like $\sin A\cos B = \frac{1}{2}(\sin(A-B)+\sin(A+B))$, etc., for dealing with $\sin(mx)\cos(nx)$ in an integrand (page 476). (Problems 68 and 70 in 7.2 are nice for future applications.)

6. Trigonometric substitution works when the function is one-to-one on the domain of integration. This is why in the table on page 478 they give bounds for the angles — the angles may not leave these ranges! Think about why the substitution $x = a\cos\theta$ (not listed) is OK so long as $0 \le \theta \le \pi$.

7. Integration by partial fractions. Review long division, that polynomials can *always* be factored into products of linear $ax+b$ and irreducible[1] quadratic ax^2+bx+c, and the form partial fractions takes, and what happens when there are repeated factors in the denominator. Remember that the quadratic term is dealt with by completing the square.

[1] A quadratic is irreducible if its discriminant b^2-4ac is negative, since then it has no roots. If it had roots, it could factor into linear terms, and if it could factor into linear terms it would have roots.

8. Problem 59 in 7.4, which is substituting $t = \tan(x/2)$ when there are rational functions of trigonometric functions, seems like a useful technique. Elsewhere it is called "magic substitution."

9. Section 7.5 describes heuristics for choosing integration strategies. (Note, the end of the section, "Can We Integrate All Continuous Functions?", is misleading. All continuous functions *do* in fact have an integral/antiderivative. In particular, it is $F(x) = \int_a^x f(t)\,dt$, where a is in the domain of f. Remember, this is just a limit [and continuous by FTC1 paired with the fact that functions which are differentiable are continuous]. Whether this can be written in terms of elementary functions is another story.)

10. Sometimes multiplying things out makes an integral easier to compute. Sometimes you are just integrating a polynomial (what is $\int \frac{x^2+2x+1}{x+1}\,dx$?).

11. Do not forget: all of the integration techniques are just techniques. The way you know you have an antiderivative is just by taking a derivative.

12. The left endpoint rule gives a lower bound for concave up functions, where the right endpoint rule gives an upper bound. There is something similar to say for concave down functions.

13. Error bounds for the Trapezoidal Rule and Midpoint Rules are similar in flavor to Taylor's Inequality: If K is an upper bound for $|f''(x)|$ for all x satisfying $a \le x \le b$, then if E_T and E_M are the errors for the trapezoidal and midpoint rules (respectively),
$$|E_T| \le \frac{K(b-a)^3}{12n^2} \qquad\qquad |E_M| \le \frac{K(b-a)^3}{24n^2}$$
Simpson's rule has a similar error bound (page 514).

14. Problem 47 of Section 7.8 gives that for a concave down positive function f, $T_n < \int_a^b f(x)\,dx < M_n$. These are likely better bounds than the left endpoint and right endpoint bounds.

15. Integrals can be improper for two reasons: one of the bounds of integration is ∞, or the integrand is discontinuous at some point. Both cases require a limit.

16. Integrals which are improper at more than one place must be first split up into two integrals. For instance, $\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_a^{\infty} f(x)\,dx$.

17. Check $\int_1^\infty \frac{1}{x^p}\,dx$ converges if and only if $p > 1$.[2] What does this say about $\int_0^1 \frac{1}{x^p}\,dx$? Does $\int_0^\infty \frac{1}{x^p}\,dx$ ever converge?

18. Like for series, improper integrals have a comparison theorem. Comparing integrands, knowing whether one integral converges or diverges sometimes lets you see whether the other does. (Non-example: since $\frac{1}{x^2} \le 1$ for all $x \ge 1$, since $\int_1^\infty \frac{1}{x^2}\,dx$ converges, so does $\int_1^\infty 1\,dx$.)

[2] This was the fact that showed $\sum_{n=0} \frac{1}{n^p}$ converges if and only if $p > 1$. Since the integral converges, by the integral test the sum converges. Kind of backwards to learn that now, right?

1.2 Further Applications of Integration (Chapter 8)

1. Arc length is defined to be the limit of the sum of the lengths of a secant-line-approximation of a curve as the number of secant lines goes to infinity.

2. This is difficult to compute, so the book proves that this limit is equal to $\int_a^b \sqrt{1+f'(x)^2}\,dx$ if the derivative of f is continuous. Mnemonic: $ds = \sqrt{dx^2+dy^2}$.

3. If $x = g(y)$ and g is one-to-one, then the arc length can also be computed with $\int_c^d \sqrt{1+g'(y)^2}\,dy$.

4. Like most things, one can make a function out of arc length by taking the arc length between some varying endpoints.

5. Area of surfaces of revolution comes from approximating a surface by the outsides of frusta (cut cones). The area element is $dA = 2\pi r\,ds$, where ds is the arc length element. So, $S = \int_a^b 2\pi f(x)\sqrt{1+f'(x)^2}\,dx$ or $\int_c^d 2\pi g^{-1}(y)\sqrt{1+g'(y)^2}\,dy$.

6. For hydrostatic pressure, the pressure $\rho g d$ has d being the vertical distance to the surface of the liquid. Liquids are only in equilibrium when all surfaces for connected regions of liquid are level with each other. Force is the integral of presure over some area (so an integral of an integral).

7. Center of mass is basically the average position of all the mass. Take all the x coordinates and weight them by density at that point, then sum them up and divide by total density. That gives $\overline{x}$, the center of mass in the x direction. Supposing a shape has constant density, then one can compute

$$\overline{x} = \frac{\int_a^b x f(x)\,dx}{\int_a^b f(x)\,dx}.$$

   In the y direction, either rotate the whole shape ninety degrees, or find the average y position by averaging centers of a bunch of vertical strips, weighted by the length of the strip:

$$\overline{y} = \frac{\frac{1}{2}\int_a^b f(x)^2\,dx}{\int_a^b f(x)\,dx}$$

8. For more complicated shapes, the moments may be added or subtracted to add or remove regions.

9. It might be useful to know the Theorem of Pappus: a volume of revolution's volume may be calculated by multiplying the area of the region being revolved with the distance its centroid travels.

10. Probability distributions are functions such that $\int_{-\infty}^{\infty} f(x)\,dx = 1$. Expected (average) value is center of mass: $\int_{-\infty}^{\infty} x f(x)\,dx$ the average value of the distribution f.

1.3 Infinite Sequences and Series (Chapter 11)

1. A power series is a function of the form $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ for a bunch of constants c_n and some constant a (it's "center").

2. Relevant data for a power series: radius of convergence and interval of convergence. (The second is another name for the domain of the function.)

3. Theorem: if a power series centered at a converges at $a+b$, then it converges for all x satisfying $|x-a| < |b|$.

4. Term-by-term differentiation and integration keep the same radius of convergence.

5. Know the power series for $\frac{1}{1-x}$. Remember that you can take derivatives or integrate it to get other series. Remember that you can replace the x with whatever by substitution to get another power series (and know what that does to the radius of convergence).

6. Intuition for radius of convergence: it is the largest radius around the center which does not include discontinuities ("poles").

7. (Taylor series). If f has a power series representation at a (i.e., there is some R such that whenever x satisfies $|x-a| < R$ then $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$), then the coefficients are equal to $c_n = \frac{f^{(n)}(a)}{n!}$.

8. This means that if you have a power series which equals a particular function, then the derivatives can be computed directly from the coefficients ($f^{(n)}(a) = n!c_n$).

9. This also means that a series for f can sometimes be written as $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$ (this is called "the Taylor series for the function"). However, this does not always equal the original function. The radius of convergence might be smaller than the domain of f. For instance, $\frac{1}{1-x}$ has the domain all real numbers except 1, but its series representation at 0 has a radius of convergence equal to 1. There are other functions which give a Taylor series with a radius of convergence which is only 0, for instance the function $f(x) = e^{-1/x^2}$ with $f(0) = 0$ (this is problem 74 in 11.10).

10. When $a = 0$ it may be called a Maclaurin series rather than a Taylor series, if you want.

11. The Taylor polynomial T_n is $T_n(x) = \sum_{n=0}^{n} \frac{f^{(n)}(a)}{n!}(x-a)^n$. This is called the "$n$th-order approximation of f." The Taylor series is $T(x) = \lim_{n\to\infty} T_n(x)$.

12. The Taylor remainder is $R_n(x) = f(x) - T_n(x)$. It represents the difference between the nth-order Taylor polynomial and the actual function.

13. If $\lim_{n\to\infty} R_n(x) = 0$ for all x satisfying $|x-a| < R$, then $f(x)$ is equal to its Taylor series for those x. This is because then $\lim_{n\to\infty} f(x) = \lim_{n\to\infty}(T_n(x) + R_n(x)) = \lim_{n\to\infty} T_n(x) = T(x)$.

14. Taylor's Inequality. Suppose M is a number and d is a number such that $0 < d < R$. If $|f^{(n+1)}(x)| \leq M$ for all x satisfying $|x-a| \leq d$ (i.e., M is an upper bound for $|f^{(n+1)}(x)|$ for all x at most d away from a), then we know that for all x satisfying $|x-a| \leq d$ we have $|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$.

15. When you are trying to show a function equals its Taylor series, then use the Inequality to bound the remainder, use the squeeze theorem to show the remainder goes to zero as $n \to \infty$, and then use the above fact about the function equaling its Taylor series when this limit is zero.

16. This is a sidebar from the chapter: the remainder term is equal to $R_n(x) = \frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)\,dt$. This might be useful if you want something a bit better than just a bound.

17. Also from the sidebar: Like the mean value theorem, there exists a number z between x and a such that $R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-a)^{n+1}$.

18. Reminder: An upper bound for a set is a number which is at least as large as everything in the set. Bounds might be tight, in that it is the least of all upper bounds (these exist due to the completeness axiom, though they might be hard to calculate). Bounds might be loose, in that there might be a tighter one, though many times bounds do not need to be particularly tight to solve a problem.

19. One useful loose bound: for a series, taking the absolute values of the terms gets a series with a value (if it converges) at least as large as the original series. Removing large numbers from the denominator gets a series (if it converges) which is at least as large as the original. If you get to a geometric series, it is easy to compute.

20. $\lim_{n\to\infty} \frac{x^n}{n!} = 0$ for all x. (This is true because $\sum x^n/n!$ converges by the ratio test, and so the limit of the terms is 0 by the divergence test [if it weren't zero, it wouldn't converge].)

21. The book on page 757 shows to prove e^x equals its Taylor series for all x.

22. Binomial series: for any k and any x satisfying $|x| < 1$, then $(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n$.

23. To calculate $\binom{k}{n}$: three cases are a) $\binom{k}{0} = 1$, b) $\binom{k}{1} = k$, and c) $\binom{k}{n} = \frac{k}{n}\binom{k-1}{n-1}$. This means $\binom{k}{n} = \frac{k}{n} \cdot \frac{k-1}{n-1} \cdot \cdots \cdot \frac{k-n+2}{2} \cdot \frac{k-n+1}{1}$.

24. Table 1 on page 762 of chapter 11.10 gives various Taylor series and their radii of convergence. Know how to compute them yourself if needed. It is best to also know how to prove that the functions equal their Taylor series on the radius of convergence, too.

25. Limits can be computed using a Taylor series.

26. The Taylor Inequality can be used to estimate the value of a function to within a desired precision.

27. Series can be multiplied and divided. When multiplying series (remember, this is long multiplication, *not* term-by-term multiplication), the radius of convergence ends up being the smaller of the two radii of convergence. When dividing, it is less clear. (It is unlikely you will have to divide series.)

2 Problems

Previous homework problems are good to look at. Here are some others I could come up with. Problems plus has other interesting problems.

1. Find some a so that $\int_a^\infty \frac{1}{x^2+1}\,dx < 0.001$. Intended method: reinterpret this as $\int_0^{1/a} \frac{1}{x^2+1}\,dx$ and then do term-by-term integration of a series, then find a geometric series for an upper bound.

2. Use a Taylor series to find the 20th derivative of $\frac{1}{(1-x)^2}$ at $x = 0$.

3. Let $f(x)$ be a function with a Taylor series centered at $a = 0$ that has a non-zero radius of convergence. Show that $g(x) = \frac{d^{50}}{dx^{50}} f(x^2)$ has a critical point at 0.

4. Show that $\sin x$ is equal to its Taylor series centered at $a = 0$, for all x.

5. A generalization of this is: Show that if f is a function such that $f^{(n)}(x)$ exists for all x and n, and such that there is some constant c such that $|f^{(n)}(x)| \leq c$ for all x and n, then f is equal to its Taylor series.

6. Find the ratio b/a such that the center of mass of the region in the first quadrant between a circle of radius b and a circle of radius a lies on the circle of radius a.

3 Solutions

I am experimenting with a style where each fact is given its own list entry, and subordinate facts are put into sublists, with the hope that no relevant fact goes unstated.

1. (a) The antiderivative of $\frac{1}{x^2+1}$ is $\tan^{-1} x$.
   (b) Hence the integral $\int_a^\infty \frac{1}{x^2+1}\,dx$ is $\lim_{b\to\infty}(\tan^{-1} b - \tan^{-1} a)$.
   (c) The tangent function has an asymptote at $\frac{\pi}{2}$, so $\lim_{b\to\infty} \tan^{-1} b = \frac{\pi}{2}$.
   (d) Thus, the integral is $\frac{\pi}{2} - \tan^{-1} a$.
   (e) By looking at triangles (or some identities), this is equal to $\tan^{-1} \frac{1}{a}$.
   (f) Hence the integral is $\int_0^{1/a} \frac{1}{1+x^2}\,dx$.
   (g) The series for $\frac{1}{1+x^2}$ at 0 is $\sum_{n=0}^\infty (-1)^n x^{2n}$.
   (h) The integral of this series is $\sum_{n=0}^\infty \frac{(-1)^n x^{2n+1}}{2n+1}$.
   (i) Hence, $\int_0^{1/a} \frac{1}{1+x^2}\,dx = \sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)a^{2n+1}}$.

(j) The terms of this series are bounded above by $\frac{1}{a^{2n+1}}$ since $2n+1 \geq 1$.

(k) So the series itself is bounded above by $\sum_{n=0}^{\infty} \frac{1}{a^{2n+1}}$, which is geometric, so equals $\frac{1/a}{1-a^2} = \frac{a}{a^2-1}$.

(l) To get this to be smaller than 0.001, we must solve $0.001 > \frac{a}{a^2-1}$, which is equivalently $a^2 - 1000a - 1 > 0$.

(m) The larger root of $a^2 - 1000a - 1$ is $\frac{1000+\sqrt{1000^2+4}}{2}$. Since $\sqrt{1000^2 + 2^2} < 1000 + 2$ (think of a triangle's hypotenuse), this value has an upper bound of $\frac{1000+1000+2}{2} = 1001$.

(n) Hence, if $a \geq 1001$, $\int_0^{1/a} \frac{1}{1+x^2}\,dx < 0.001$.

(o) Therefore, if $a \geq 1001$, $\int_a^{\infty} \frac{1}{1+x^2}\,dx < 0.001$.

(p) (In fact, $\frac{1}{2}\pi - \tan^{-1} 1001 < 0.0009990007$. An exact answer is $a = \tan(\frac{1}{2} - 0.001) < 999.9997$.)

2. (a) Notice $\frac{d}{dx}\frac{1}{1-x} = \frac{1}{(1-x)^2}$.

(b) The Taylor series for $\frac{1}{1-x}$ is $\sum_{n=0}^{\infty} x^n$.

(c) Using term-by-term differentiation, the derivative of $\frac{1}{1-x}$ has as its Taylor series $\sum_{n=1}^{\infty} nx^{n-1}$. (Note the constant term has a derivative of 0, so the lower bound of the sum becomes 1.)

(d) This is the Taylor series of $\frac{1}{(1-x)^2}$.

(e) The x^n term of a Taylor series is the coefficient $\frac{f^{(n)}(0)}{n!}$.

(f) Since the x^n term of the Taylor series has the coefficient $n+1$, we have the equations $n + 1 = \frac{f^{(n)}(0)}{n!}$ for each $n \geq 0$.

(g) So solving, $f^{(n)}(0) = (n+1)!$.

(h) The 20th derivative at 0 is $21!$.

3. (a) The Taylor series of $f(x)$ is $\sum_{n=0}^{\infty} \frac{f^{(n)} x^n}{n!}$.

(b) A series for $h(x) := f(x^2)$ is $\sum_{n=0}^{\infty} \frac{f^{(n)} x^{2n}}{n!}$.

(c) If $R > 0$ is the radius of convergence of f, then $\sqrt{R}$ is the radius of convergence for h, since $|x^2| < R$ implies $|x| < \sqrt{R}$.

(d) Thus this series *is* a Taylor series of h, centered at 0.

(e) A critical point of $g(x)$ is a place where $g'(x)$ is zero or does not exist.

(f) The derivative of g is the 51st derivative of $h(x)$.

(g) The 51st derivative of a function shows up in the coefficient to x^{51} as $\frac{h^{(51)}(0)}{51!}x^{51}$.

(h) Since the series for h has no odd powers of x, the coefficient for x^{51} is zero.

(i) Thus $h^{(51)}(0) = 0$.

(j) Therefore $g'(0) = 0$, so it is indeed a critical point of g.

4. (a) Let $f(x) = \sin x$.

   (b) The derivatives of f are

n	$f^{(n)}(x)$
0	$\sin x$
1	$\cos x$
2	$-\sin x$
3	$-\cos x$
4	$\sin x$
...	...

   (c) For all x, since $|\pm \sin x| \leq 1$ and $|\pm \cos x| \leq 1$, we see that $|f^{(n)}(x)| \leq 1$ for all x and n.

   (d) The terms of the Taylor series for $\sin x$ then satisfy $|\frac{f^{(n)}(0)x^n}{n!}| \leq |\frac{x^n}{n!}|$.

   (e) For all x, $\lim_{n\to\infty} \frac{|x^{n+1}/(n+1)!|}{|x^n/n!|} = \lim_{n\to\infty} \frac{|x|}{n+1} = 0 < 1$.

   (f) By the ratio test, $\sum_{n=0}^{\infty} |\frac{x^n}{n!}|$ converges for all x.

   (g) By the comparison test, $\sum_{n=0}^{\infty} |\frac{f^{(n)}(0)x^n}{n!}|$ converges for all x.

   (h) Absolute convergence implies convergence, so $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)x^n}{n!}$ converges for all x.

   (i) Therefore the Taylor series for f has the same domain as f does (i.e., the Taylor series has an infinite radius of convergence).

   (j) For every $d > 0$: [3]

      i. Let $M = 1$.

      ii. We have $|f^{(n+1)}(x)| \leq 1 = M$ for all x satisfying $|x| \leq d$.

      iii. By the Taylor inequality, $|R_n(x)| \leq \frac{M|x|^{n+1}}{(n+1)!} = \frac{|x|^{n+1}}{(n+1)!}$.

      iv. Since $\sum_{n\to\infty} \frac{|x|^n}{n!}$ converges, $\lim_{n\to\infty} \frac{|x|^n}{n!} = 0$. (Otherwise, by the divergence test the sum would diverge.)[4]

      v. So $\lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0$.

      vi. By the squeeze theorem, $\lim_{n\to\infty} |R_n(x)| = 0$ for all x satisfying $|x| \leq d$.

      vii. Thus $\lim_{n\to\infty} R_n(x) = 0$ by continuity of absolute value.

      viii. Since $R_n(x) = \sin x - T_n(x)$, $\lim_{n\to\infty}(\sin x - T_n(x)) = 0$.

      ix. $\lim_{n\to\infty} T_n(x) = \lim_{n\to\infty} \sum_{m=0}^{n} \frac{f^{(m)}(0)x^m}{m!} = \sum_{m=0}^{\infty} \frac{f^{(m)}(0)x^m}{m!}$.

      x. Therefore $\sin x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)x^n}{n!}$ for all $|x| \leq d$.

   (k) Since d may be arbitrarily large, $\sin x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)x^n}{n!}$ for all x.

   (l) Two functions are equal if they have the same domains and the same values for each input, so therefore $\sin x$ is equal to its Taylor series for all x. $\square$

[3] Talking about d is a technicality of the Taylor inequality

[4] I like this argument because it uses series to prove something about a limit. It is akin to interpreting a limit as an integral and then using an antiderivative to evaluate it. One could try showing the limit is zero directly, but I have a taste for the indirect argument.

5. This is very similar to problem 4. In fact, problem 4 is when $c = 1$.

6. (a) First, to make sure this can even happen, think about this: if a is very small, then the center of mass is surely *inside* the region, and if a is very close to b, then the center of mass is surely *outside* the region. Then by the intermediate value theorem, at some point the center of mass lies *on* the boundary circle.

   (b) The outer circle is given by the equation $y = \sqrt{b^2 - x^2}$. The inner circle by $y = \sqrt{a^2 - x^2}$.

   (c) The x-axis moment is the different between the moment for the b circle and the moment for the a circle (this trick is pretending that the inner circle is negative mass which cancels the mass from the outer circle).

   (d) The moment from b is $\int_0^b x\sqrt{b^2 - x^2}\,dx$.

   (e) With $u = b^2 - x^2$, $du = -2x\,dx$, so the integral is $\int_{b^2}^0 -\frac{1}{2}\sqrt{u}\,du = [-\frac{1}{3}u^{\frac{3}{2}}]_{b^2}^0 = \frac{1}{3}b^3$.

   (f) The inner circle then gives $\frac{1}{3}a^3$.

   (g) So the total moment is $\frac{1}{3}(b^3 - a^3)$.

   (h) The mass of the region is the difference of the areas of the circles, namely $\frac{1}{4}(\pi b^2 - \pi a^2)$.

   (i) Hence $\overline{x} = \frac{\frac{1}{3}(b^3 - a^3)}{\frac{\pi}{4}(b^2 - a^2)} = \frac{4(b^2+ab+a^2)}{3\pi(b+a)}$

   (j) By symmetry over the line $y = x$, the $\overline{y}$ coordinate for center of mass is equal to $\overline{x}$.

   (k) We want the center of mass to lie on the inner circle, which has the equation $x^2 + y^2 = a^2$, so $2\overline{x}^2 = a^2$, hence $\overline{x} = \frac{a}{\sqrt{2}}$.

   (l) Then the equation becomes $\frac{4(b^2+ab+a^2)}{3\pi(b+a)} = \frac{a}{\sqrt{2}}$.

   (m) Let $\lambda = \frac{b}{a}$ be the ratio between b and a.

   (n) Then the equation becomes $\frac{4(\lambda^2a^2+\lambda a^2+a^2)}{3\pi(\lambda a+a)} = \frac{a}{\sqrt{2}}$.

   (o) The a factors out and can be eliminated, giving $\frac{4(\lambda^2+\lambda+1)}{3\pi(\lambda+1)} = \frac{1}{\sqrt{2}}$.

   (p) This is equivalent to the quadratic $\lambda^2 + (1 - \frac{3\pi}{4\sqrt{2}})\lambda + (1 - \frac{3\pi}{4\sqrt{2}}) = 0$.

   (q) Since $1 - \frac{3\pi}{4\sqrt{2}} < 0$, this polynomial has real roots.

   (r) (One root is positive, one is negative. The positive one is the answer.)