# Midterm 1 Review

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- Integration by parts ($\int u\,dv = uv - \int v\,du$). (Derive this from the product rule.)
- What is a sequence? (A list of numbers $a_1, a_2, \ldots$. Sometimes written as $\{a_n\}$ or $\{a_n\}_{n=1}^{\infty}$ or $\{a_n\}_{n\in\mathbb{N}}$ or $(a_n)_n$). Give an example of a recursively defined sequence.
- Definition: A sequence $\{a_n\}$ has a *limit* $L$ (written $\lim_{n\to\infty} a_n = L$) if

  1. for every $\varepsilon > 0$,
  2. there is an integer $N$ such that
  3. for every $n > N$
  4. $|a_n - L| < \varepsilon$.

  In other words, a sequence has a limit $L$ if for any error $\varepsilon > 0$ you may tolerate, there is a point after which the sequence is always within $\varepsilon$ of $L$.

  This is defining a limit of *sequences*, rather than the "continuous" limit from Math 1A. It is slightly different.

- Theorem: Let $f$ be some function, and let $\{a_n\}$ be a sequence where $a_n = f(n)$. If $\lim_{x\to\infty} f(x) = L$ (this is a "continous" limit), then $\lim_{n\to\infty} a_n = L$ (this is a sequence limit).

  In other words, you can treat sequence limits as continuous limits if the inside is a function which isn't just defined on integers.

- What is the definition of $\lim_{n\to\infty} a_n = \infty$? (Hint for more formal definition: a sequence tends toward infinity if for any $M$, the sequence is eventually always larger than $M$.)
- Limit laws: addition, subtraction, multiplication, constants, quotients, and powers work as you would expect for sequence limits.
- Squeeze theorem: works for sequences, too.
- Theorem: if $\lim_{n\to\infty}|a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$. (Why?)
- Theorem: if $f$ is a function continuous at $L$ and $\lim_{n\to\infty} a_n = L$, then $\lim_{n\to\infty} f(a_n) = f(L)$. ("Substitution.")
- Write the definitions for an *increasing* sequence, a *decreasing* sequence, (and *monotonic*), *bounded above*, *bounded below*, (and *bounded sequence*).
- What does the Monotonic Sequence Theorem say? How does it use the Completeness Axiom of the real numbers?

- Definition: an *(infinite) series* for a sequence $\{a_n\}$ is written as $\sum_{n=1}^{\infty} a_n$, which is shorthand for $\lim_{m\to\infty} \sum_{n=1}^{m} a_n$.

  More formally, a series is the limit of the sequence of partial sums $\{\sum_{n=1}^{m} a_n\}_{m=1}^{\infty}$. Sometimes $\sum_{n=1}^{m} a_n$ is notated as $s_m$ and $\sum_{n=1}^{\infty} a_n$ as $s$.

- Definition: a series *converges* if the sequence of partial sums converges (to the *sum* of the series). A series *diverges* if it doesn't converge. (What is a series which diverges but does not tend to infinity?)

- What is a geometric series? What is the condition for a geometric series to converge? to diverge? What is the sum of a geometric series when it converges?

- The harmonic series: does it converge?

- Theorem (Test for Divergence): If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$. (So, if the limit is *not* zero or does not exist, the series is *divergent*.)

- Series laws: you can add and subtract convergent series to get another convergent series, and you can multiply a convergent series by a constant to get another convergent series. (But you cannot multiply two convergent series term-by-term to get a convergent series, for instance $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ with itself.)

- Theorem (The Integral Test). Suppose $f$ is a continuous, positive, decreasing function on $[1, \infty)$, and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty}$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)\,dx$ is convergent. (Remember: this is shorthand for $\lim_{b\to\infty} \int_1^b f(x)\,dx$.)

- What does "if and only if" mean?

- What is a $p$-series? When do they converge? when do they diverge? (How do you use the Integral Test to show this?)

- Definition: the *remainder* is the difference between the sum of a series and one of its partial sums. Notation: $R_n = s - s_n$. Why might we care about the remainder?

- An estimate for the remainder: if $f$ and $a_n$ are like those in the Integral Test, $\int_{n+1}^{\infty} f(x)\,dx \le R_n \le \int_n^{\infty} f(x)\,dx$. (Draw some pictures of rectangles to prove this to yourself.)

- Theorem (The Comparison Test): If $a_n$ and $b_n$ are two sequences such that $0 \le a_n \le b_n$, then

  1. If $\sum b_n$ converges, $\sum a_n$ converges, too; and
  2. If $\sum a_n$ diverges, then $\sum b_n$ diverges, too.

- Theorem (The Limit Comparison Test): If $\sum a_n$ and $\sum b_n$ are two series with positive terms, and if $0 < \lim_{n\to\infty} \frac{a_n}{b_n} < \infty$ exists, then either both series converge or both series diverge.

In fact, the proof can be adapted to saying something stronger, that if $\frac{a_n}{b_n}$ is eventually bounded below and above by a pair of positive numbers, then both series either converge or diverge.

- If a series $\sum a_n$ converges by the comparison test against a convergent series $\sum b_n$, then remainders for $\sum a_n$ can be estimated by using remainders for $\sum b_n$. Useful if remainders for $\sum a_n$ are difficult to compute/estimate.
- An *alternating series* is a series whose terms are alternately positive and negative.
- Theorem (Alternating Series Test): If $\sum a_n$ is an alternating series where $b_n = |a_n|$ is monotonically decreasing and $\lim_{n\to\infty} b_n = 0$, then the series $\sum a_n$ is convergent. (This says *nothing* about whether an alternating series diverges; another test is necessary.)
- Theorem (Alternating Series Estimation): If $\sum a_n$ is a convergent alternating series, then $|R_n| \leq |a_{n+1}|$. In other words, the difference between $\sum_{i=1}^{n} a_i$ and $\sum_{i=1}^{\infty} a_i$ for a convergent alternating series is at most (plus or minus) the next term $a_{n+1}$.
- Definition: a series $\sum a_n$ is *absolutely convergent* if $\sum |a_n|$ is convergent. Theorem: all absolutely convergent series are also convergent. Definition: a series is *conditionally convergent* if it is convergent but not absolutely convergent. What are examples of absolutely and of conditionally convergent series?
- Theorem (The Ratio Test): If $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = L$ and
  1. $L < 1$, then $\sum a_n$ is absolutely convergent;
  2. $L > 1$ or is infinite, then $\sum a_n$ is divergent; or
  3. $L = 1$, then the series is either convergent or not convergent. (Examples both ways?)

  In the proof for when $L < 1$, they choose some $r$ such that $L < r < 1$ and use the comparison test against a geometric series with that $r$. This could be useful to keep in mind.

  In fact, the theorem can be adapted to saying that if $|\frac{a_{n+1}}{a_n}|$ is eventually bounded above by a number less than 1, then $\sum a_n$ is absolutely convergent. (This is true of the root test as well.)
- Theorem (The Root Test): This is similar to the ratio test, but now $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$. (What is a proof?)
- What do the ratio and root tests say about $p$-series? About geometric series?
- Section 11.7 has a list of heuristics for choosing a test.
- For each test, (try to) come up with examples 1) which satisfy the test for each possible conclusion, and which 2) do not satisfy the test, but still converge/diverge.
- Try to make a map of some kind for dividing the Land of Series into Those that Converge and Those that Diverge.