The log test

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Thinking about how the ratio and root tests work, I realized that a similar test could be devised for comparing against p-series rather than geometric series. Here is the test:

Theorem 1. *Let* $\sum_{n=1}^{\infty} a_n$ *be a series of positive terms. Consider* $\lim_{n\to\infty} \log_n a_n^{-1}$*. Then:*

1. *If this limit diverges to positive infinity or to a number* L *greater than* 1*, then* $\sum_{n=1}^{\infty} a_n$ *converges; and*
2. *If this limit diverges to negative infinity or to a number* L *less than* 1*, then* $\sum_{n=1}^{\infty} a_n$ *diverges; and*
3. *Otherwise the test is inconclusive.*

The proof is fairly similar to that of the ratio and root tests in Stewart:

Proof. First, suppose $\lim_{n\to\infty} \log_n a_n^{-1}$ diverges to positive infinity or to a number L greater than 1. Then there is a number p such that $1 < p$ and, if the limit converges, $p < L$. There is some N such that whenever $n > N$, $\log_n a_n^{-1} > p$ (that is, the sequence of $\log_n a_n^{-1}$ is eventually always larger than p). Observe this means $a_n^{-1} > n^p$, and so $a_n < \frac{1}{n^p}$. Since $1 < p$, $\sum_{n=1} \frac{1}{n^p}$ is convergent, and so by the comparison test (starting at $n = N + 1$), $\sum_{n=1}^{N} a_n$ converges, too.

The second part is similar, but p is chosen to be less than 1, and so by the comparison test with a divergent p-series, the series must also diverge. □

Examples:

1. $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^{\ln n}}$. We compute $\log_n a_n^{-1} = \frac{\ln(n)\ln(\ln n)}{\ln n} = \ln(\ln n)$, which tends to ∞ as $n \to \infty$, so by the theorem the series converges.
2. For which p does $\sum_{n=2}^{\infty} \frac{1}{n^p \ln n}$ converge? We compute $\frac{\ln(n^p \ln n)}{\ln n} = p + \frac{\ln(\ln n)}{\ln n}$, which tends to p as $n \to \infty$, so the series converges if $p > 1$ and diverges if $p < 1$. When $p = 1$, then we may instead use the integral test.

The test is worded as if it were the ratio or root test in Stewart. It can be strengthened into the following:

Theorem 2. *Let* $\sum_{n=1}^{\infty} a_n$ *be a series of positive terms. If the sequence* $\log_n a_n^{-1}$ *is eventually bounded below by a number* $L > 1$*, then the series converges, and if the sequence is eventually bounded above by a number* $L < 1$*, then the series diverges.*