Homogeneous linear differential equations with constant coefficients

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Saturday, 21 Nov 2015

This note goes into a bit more detail than in the textbook about what is going on with homogeneous linear differential equations with constant coefficients. In other words, differential equations which look like

$$\frac{d^n y}{dt^n} + c_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \cdots + c_1\frac{dy}{dt} + c_0 = 0. \tag{1}$$

with $c_i \in \mathbb{R}$. The book talks in particular about the case $\frac{d^2y}{dt^2} + c_1\frac{dy}{dt} + c_0 = 0$, but the general case is more interesting and, in my opinion, more illuminating.

First, let us talk a bit about algebra. We have seen before in Math 1A that the derivative is *linear*, which means for f, g differentiable functions and $c \in \mathbb{R}$,

$$\frac{d}{dt}(cf) = c\frac{df}{dt} \qquad\qquad \frac{d}{dt}(f+g) = \frac{df}{dt} + \frac{dg}{dt},$$

which, using the new word of the week, means that linear combinations are transformed into linear combinations:

$$\frac{d}{dt}(c_1f_1 + c_2f_2 + \cdots + c_nf_n) = c_1\frac{df_1}{dt} + c_2\frac{df_2}{dt} + \cdots + c_n\frac{df_n}{dt}.$$

This is an important property. Linear transformations like this are the main subject of *linear algebra*. Do not miss the fact that this is saying that whenever $f_1, \dots, f_n$ are differentiable functions, then $c_1f_1 + c_2f_2 + \cdots + c_nf_n$ is a differentiable function, too. In fact, if $f_1, \dots, f_n$ are solutions to a homogeneous differential equation, then so is $c_1f_1 + c_2f_2 + \cdots + c_nf_n$.

What is the derivative anyway? A profitable point of view is that it is a function which takes a differentiable function and transforms it into another function (its derivative). Given function-transforming functions T_1 and T_2, it is also the case that $c_1T_1 + c_2T_2$ is a function-transforming function by the rule $(c_1T_1 + c_2T_2)f = c_1T_1(f) + c_2T_2(f)$. Also, $(T_1T_2)f = T_1(T_2(f))$. If $T_1 = \frac{d}{dt}$ and $T_2 = \frac{d^2}{dt^2}$, then we have $(c_1\frac{d}{dt} + c_2\frac{d^2}{dt^2})f = c_1\frac{df}{dt} + c_2\frac{d^2f}{dt^2}$. This may seem abstract, but then we can rewrite the differential equation 1 as

$$\left(\frac{d^n}{dt^n} + c_{n-1}\frac{d^{n-1}}{dt^{n-1}} + \cdots + c_1\frac{d}{dt} + c_0\right)y = 0.$$

With the rule that $\frac{d}{dt}\frac{d^k}{dt^k} = \frac{d^{k+1}}{dt^{k+1}}$ (the derivative of the kth derivative of something is the $(k+1)$th derivative of it), this can factor with complex numbers $\lambda_i \in \mathbb{C}$ as

$$\left(\frac{d}{dt} - \lambda_1\right) \dots \left(\frac{d}{dt} - \lambda_n\right) y = 0.$$

The reason it can factor is the fundamental theorem of algebra: every polynomial has at least one root, which has the consequence (due to long division) that every polynomial can factor into a product of linear terms.

An interesting property of the linear factors is that they commute:

$$\left(\frac{d}{dt} - \lambda\right)\left(\frac{d}{dt} - \mu\right) = \frac{d^2}{dt^2} - (\lambda + \mu)\frac{d}{dt} + \lambda\mu = \left(\frac{d}{dt} - \mu\right)\left(\frac{d}{dt} - \lambda\right).$$

Suppose $\lambda_1, \dots, \lambda_r \in \mathbb{C}$ are the distinct roots in the factorization, with λ_i having multiplicity k_i, so that the differential equation can be written as

$$\left(\frac{d}{dt} - \lambda_1\right)^{k_1} \dots \left(\frac{d}{dt} - \lambda_r\right)^{k_r} y = 0. \tag{2}$$

The commutativity has the consequence that whenever y is a solution to $\left(\frac{d}{dt} - \lambda_j\right)^{k_j} y = 0$, then it is a solution to the whole differential equation, since this jth term can be brought to the end:

$$\left(\frac{d}{dt} - \lambda_1\right)^{k_1} \dots \left(\frac{d}{dt} - \lambda_{j-1}\right)^{k_{j-1}} \left(\frac{d}{dt} - \lambda_{j+1}\right)^{k_{j+1}} \dots \left(\frac{d}{dt} - \lambda_r\right)^{k_r} \left(\frac{d}{dt} - \lambda_j\right)^{k_j} y = 0,$$

where this last term will be zero due to y being a solution, and zero is a solution to the rest of the terms, so it is indeed a solution to the full differential equation.

A nice consequence of commutativity and linearity is that to solve this differential equation, we just need to solve each of these differential equations:

$$\begin{gathered}\left(\frac{d}{dt} - \lambda_1\right)^{k_1} y = 0 \\ \vdots \\ \left(\frac{d}{dt} - \lambda_r\right)^{k_r} y = 0,\end{gathered}$$

and then linear combinations of these solutions are solutions to the full differential equation. We will also be able to say that this covers *every* possible solution to the differential equation.

Thus, we will now restrict our attention to the following differential equation:

$$\left(\frac{d}{dt} - \lambda\right)^{k} y = 0.$$

where $k \in \mathbb{N}$ and $\lambda \in \mathbb{C}$. There are some clear solutions to this differential equation, for instance $y = 0$ and also $y = e^{\lambda t}$, where the latter is because $\left(\frac{d}{dt} - \lambda\right) e^{\lambda t} = \frac{d}{dt}e^{\lambda t} - \lambda e^{\lambda t} =$

$\lambda e^{\lambda t} - \lambda e^{\lambda t} = 0$. (In other words, $(\frac{d}{dt} - \lambda)y = 0$ is the differential equation $y' = \lambda y$.) But, there are in fact k *linearly independent* solutions, where $e^{\lambda t}$ is one of them.

We will show the general solution to this differential equation by induction on k. (It might be possible by using Taylor series, too, but induction lets us use integrating factors as we learned in class.)

Lemma 1. *Solutions to the differential equation* $\left(\frac{d}{dt} - \lambda\right)^k y = 0$, *with* $k \geq 1$, *are of the form*

$$y = C_0 e^{\lambda t} + C_1 t e^{\lambda t} + \cdots + C_{k-1} t^{k-1} e^{\lambda t} \tag{3}$$

for arbitrary constants $C_0, \ldots, C_{k-1}$.

Proof. We will induct on k. When $k = 1$, we have already seen that this is true. So, assume it is true for $k - 1$, and we will prove it is true for k. We can break our differential equation up like this:

$$\left(\frac{d}{dt} - \lambda\right)^{k-1} \left(\frac{d}{dt} - \lambda\right) y = 0.$$

Our induction hypothesis says that the only solutions to this are those such that $\left(\frac{d}{dt} - \lambda\right) y = C_0 e^{\lambda t} + C_1 t e^{\lambda t} + \cdots + C_{k-2} t^{k-2} e^{\lambda t}$, hence the differential equation becomes

$$\left(\frac{d}{dt} - \lambda\right) y = C_0 e^{\lambda t} + C_1 t e^{\lambda t} + \cdots + C_{k-2} t^{k-2} e^{\lambda t},$$

also written as

$$\frac{dy}{dt} - \lambda y = C_0 e^{\lambda t} + C_1 t e^{\lambda t} + \cdots + C_{k-2} t^{k-2} e^{\lambda t}.$$

Finding the integrating factor $e^{-\lambda t}$, we have

$$\frac{d}{dt}\left(e^{-\lambda t} y\right) = C_0 + C_1 t + \cdots + C_{k-2} t^{k-2}$$

so then for some constant K,

$$e^{-\lambda t} y = K + C_0 t + \frac{1}{2} C_2 t^2 + \cdots + \frac{1}{k-1} C_{k-2} t^{k-1}.$$

With this, the solution can be written as

$$y = K e^{\lambda t} + C_0 t e^{\lambda t} + \frac{1}{2} C_1 t^2 e^{\lambda t} + \cdots + \frac{1}{k-1} C_{k-2} t^{k-1} e^{\lambda t},$$

which by renaming variables (with $D_0 = K$, and $D_i = \frac{1}{i} C_{i-1}$ for $i \geq 1$) is of the required form. Hence, this completes the induction argument. $\square$

This gives us a bunch of solutions to the full differential equation 2. Are these *all* of the solutions, though? The answer is "yes," but the proof is somewhat annoying so here is a sketch. For this, we first need to do a calculation. Let $\lambda, \mu \in \mathbb{C}$ be distinct complex numbers, and let $n \geq 0$. Then, let us solve the differential equation

$$\left(\frac{d}{dt} - \mu\right) y = t^n e^{\lambda t}.$$

By finding the integrating factor $e^{-\mu t}$ and integrating by parts, we can get that the solution is of the form $y = p(t)e^{\lambda t}$, where $p(t)$ is a polynomial of degree n. So, if y is a solution to differential equation 2, we know that $z = \left(\frac{d}{dt} - \lambda_2\right)^{k_2} \dots \left(\frac{d}{dt} - \lambda_r\right)^{k_r} y$ is a solution to $\left(\frac{d}{dt} - \lambda_1\right)^{k_1} z = 0$, hence $z = C_0 e^{\lambda t} + \cdots + C_{k_1 - 1} e^{\lambda t}$. By the above calculation and by induction, we can conclude that y must actually be a linear combination of solutions of the form in equation 3.

For a second-order homogeneous linear differential equation, as you have seen in the book, we have differential equations of the following flavors:

$$\left(\frac{d}{dt} - \lambda_1\right)\left(\frac{d}{dt} - \lambda_2\right) y = 0$$

with $\lambda_1 \neq \lambda_2$, where either both are real or $\lambda_2 = \overline{\lambda_1}$,[1] and

$$\left(\frac{d}{dt} - \lambda\right)^2 y = 0$$

with $\lambda \in \mathbb{R}$.

In the second case, solutions according to our result are of the form $y = C_0 e^{\lambda t} + C_1 t e^{\lambda t}$, and in the first case $y = C_0 e^{\lambda_1 t} + C_1 e^{\lambda_2 t}$.

The book distinguishes between real solutions and complex solutions for the first case. The reason is that they want all solutions to differential equations to be real (where we have been allowing for complex-valued solutions so far!). Using Euler's identity $e^{i\theta} = \cos\theta + i\sin\theta$, they take the complex case and rewrite, where $\lambda = \alpha + \beta i$:

$$\begin{aligned}
y &= C_0 e^{(\alpha+\beta i)t} + C_1 e^{(\alpha-\beta i)t} \\
&= \left(C_0 e^{\beta i t} + C_1 e^{-\beta i t}\right) e^{\alpha t} \\
&= \left(C_0 \cos(\beta t) + C_1 \cos(-\beta t) + C_0 i \sin(\beta t) + C_1 i \sin(-\beta t)\right) e^{\alpha t} \\
&= \left((C_0 + C_1)\cos(\beta t) + (C_0 - C_1) i \sin(\beta t)\right) e^{\alpha t}.
\end{aligned}$$

The condition for this being real is for $\overline{y} = y$, which gives $\overline{C_0} + \overline{C_1} = C_0 + C_1$ and $-\overline{C_0} + \overline{C_1} = C_0 - C_1$. By adding these equations together, we see $\overline{C_0} = C_1$, so letting A be half the real part of C_0 and B being half the imaginary part of C_0, we get the textbook's general real solution

$$y = \left(A\cos(\beta t) + B\sin(\beta t)\right) e^{\alpha t} = \sqrt{A^2 + B^2} \sin\left(\beta t + \tan^{-1}\frac{A}{B}\right) e^{\alpha t}.$$

[1] These are the only two cases for roots when the differential equation has real coefficients because of the fact that the complex conjugate of a root is also a root.