Final Exam Review

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Tuesday, 8 December 2015

1 Topics

This is a short overview of topics we have covered since the last midterm. Of course, older topics are still relevant, so it would be good to review them.

1.1 Differential equations (Chapter 9)

1. A *differential equation* is an equation involving a function y along with its derivatives y', y'', etc.
2. Interpret $\frac{dP}{dt} = kP$ as a physical system, both when k is positive and when k is negative. Same for the logistic differential equation $\frac{dP}{dt} = kP(1 - \frac{P}{M})$.
3. How do you find equilibrium solutions? What does it mean for an equilibrium solution to be stable?
4. Model the motion of a spring-mass system (answer: $mx'' + kx = 0$). Solve it. Find the equation of motion when the mass is released from rest. Find the equation of motion when the mass is launched with a particular velocity from the rest length of the spring.
5. Reinterpret the problem of *integration* as the problem of solving a *differential equation*.
6. Find $y''(0)$ given $y' = xy^3$ and $y(0) = 2$, *without* solving the differential equation.
7. Recall the calculus-based graphing rules: using critical points, places where the derivative is positive, where the derivative is negative. How can these apply to the problem of quickly sketching solutions to differential equations?
8. How is this similar to direction fields?
9. Euler's method with step size h is a sequence of points (x_n, y_n) such that (x_0, y_0) is the initial condition and $(x_n, y_n) = (x_{n-1} + h, y_{n-1} + hF(x_{n-1}, y_{n-1}))$. Draw a diagram which helps you remember this (like Figure 15 on page 590).
10. When do over- and under-estimates occur with Euler's method? (What does it have to do with concavity?)
11. Separable equations: how to identify them and how to begin solving them.
12. Orthogonal trajectories: how to come up with the differential equation from the family of curves (page 597).
13. Mixing problems: how to model the dissolved matter by finding the net flows.

14. How do you deal with antiderivatives of $\frac{1}{k-y}$? (You get an absolute value: what is the case analysis? What does it physically represent? What does it have to do with equilibrium solutions?)

15. Integral equations (exercises 33-35 in 9.3). Use the fundamental theorem of calculus and take derivatives until they become differential equations.

16. Chapter 9.4 contains the solutions to different population growth models. For the logistic equation, think about that inflection point and what it means. For instance, if you know the inflection point, can you find the carrying capacity?

17. The *order* of a differential equation is the highest-order derivative which appears.

18. A *first-order linear differential equation* is a differential equation of the form $y' + P(x)y = Q(x)$ for P and Q being some functions of x (and only x). These are solved by multiplying both sides by the integrating factor $I(x) = e^{\int P(x)\,dx}$ to get $(I(x)y)' = I(x)Q(x)$, so $y = \frac{1}{I(x)} \int I(x)Q(x)\,dx$. (Notice this is like an anti-product-rule.) Be careful with the $+C$.

19. The equation might not look like a linear differential equation. For instance, in exercise 23 of 9.5 (the Bernoulli differential equation), there is a clever substitution to get the right form.

20. What is the relationship between a *limiting velocity* (or *terminal velocity*) and equilibria?

21. What are *phase planes* and *phase trajectories*? How do you sketch solutions given a phase trajectory? What is the limitation in representing the time axis?

22. Page 632 exercise 25 looks nice (length of a caternary).

23. A differential equation doesn't involve x is called *autonomous*. For instance, $y' = y+1$. Solutions are always "time-invariant:" as in, if f is a solution, then so is $g(x) = f(x+a)$ for every a.

1.2 Second-order differential equations (Chapter 17)

1. A *second-order linear differential equation* has the form $y'' + Q(x)y' + R(x)y = G(x)$ for some functions Q, R, G. If $G(x) = 0$ for all x, then this is called *homogenous*, and otherwise *nonhomogeneous*.

2. If y_1 and y_2 are both solutions to a linear homogeneous equation, then any linear combination of y_1 and y_2 is also a solution. That is, $y = c_1y_1 + c_2y_2$ is a solution for all $c_1, c_2 \in \mathbb{R}$.

3. Two functions are *linearly independent* if they are not scalar multiples of each other.

4. If y_1 and y_2 are linearly independent solutions to a homogeneous second-order linear differential equation, then every solution is a linear combination of y_1 and y_2. (So, $y = c_1 y_1 + c_2 y_2$ covers every possible solution.)

5. For $a, b \in \mathbb{R}$, you can solve $y'' + ay' + by = 0$ by examining the roots r_1, r_2 of the *characteristic equation* $r^2 + ar + b = 0$. If they are real roots, then the solution is $y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$. If $r_1 = r_2$ is a double root, then the solution is $y = c_1 e^{r_1 x} + c_2 x e^{r_1 x}$. If r_1 and r_2 are complex roots, they are $\alpha \pm \beta i$, and the (real) solution is $y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$.

6. Solving an initial value problem for these amounts to finding c_1 and c_2 by making a system of two equations, one involving the derivative.

7. Solving a boundary value problem for these amounts to finding c_1 and c_2 by making a system of two equations, both coming from y evaluated at some point. Warning: a solution is not guaranteed to exist.

8. Nonhomogeneous second-order linear equations with constant coefficients are solved by finding one particular solution, then adding the homegenous solutions from the *complementary equation*.

9. The method of undetermined coefficients is one way of possibly finding a particular solution. The guess should not be a solution to the complementary equation (since it won't work). A way to get around this sometimes is to multiply the guess by x.

10. Don't forget that if either sin or cos show up, then both show up in the guess, with their own polynomials.

11. The method of variation of parameters involves finding functions u_1 and u_2 and two solutions y_1 and y_2 of the complementary equation such that $0 = u_1' y_1 + u_2' y_2$ and $y = u_1 y_1 + u_2 y_2$ (don't forget that y_1 and y_2 solve the complementary equation).

12. We can precompute much of the method for the equation $y'' + ay' + by = G(x)$. In fact, one should check that this is solving $0 = u_1' y_1 + u_2' y_2$ and $u_1' y_1' + u_2' y_2' = G(x)$. And, (check this) you can solve these using

$$u_1 = \int \frac{y_2 G(x)}{y_1' y_2 - y_1 y_2'}\,dx \qquad\qquad u_2 = \int \frac{y_1 G(x)}{y_1 y_2' - y_1' y_2}\,dx,$$

    Just make sure you don't forget to add constants after integrating, and then solving for the constants.

13. Damping force as used in the book is proportional to velocity. That way you get a second-order linear differential equation.

14. Forced vibrations are modeled using a nonhomogeneous linear differential equation. The homogeneous components to the solution represent the initial conditions of the system, and, loosely speaking, "taking the limit to infinity" (assuming the real part is negative) lets these *transients* decay to zero.

15. A solution to a differential equation might have a Taylor series at a point a. So, it is natural to try solving a differential equation letting $y = \sum_{n=0}^{\infty} c_n (x-a)^n$ and solving for the coefficients c_n. Perhaps you will be able to identify which elementary functions the series is for by the end. The method is to look at both sides of the equation and locate the corresponding coefficients for each power of x, since they have to be equal.

2 Problems

1. Solve the differential equation $(x-3)y' + 2y = 0$ by recognizing it as a separable differential equation.
2. Solve the differential equation $(x-3)y' + 2y = 0$ by recognizing it as a linear differential equation.
3. Solve the differential equation $(x-3)y' + 2y = 0$ using a series.

3 Solutions

2. Dividing both sides by $x - 3$, we get $y' + \frac{2}{x-3}y = 0$. The integrating factor is $\exp(\int \frac{2}{x-3}\,dx) = \exp(2\ln|x-3|) = (x-3)^2$. Muliplying both sides by this, we get $((x-3)^2 y)' = 0$, so $(x-3)^2 y = C$, and $y = \frac{C}{(x-3)^2}$.

3. Suppose a solution is of the form $y = \sum_{n=0} a_n x^n$. First, a derivative,

$$y' = \sum_{n=1} n a_n x^{n-1}.$$

Then, multiplied by $x - 3$,

$$\begin{aligned}(x-3)y' &= (x-3)\sum_{n=1} n a_n x^{n-1} \\ &= \sum_{n=1} n a_n x^n - \sum_{n=1} 3 n a_n x^{n-1}.\end{aligned}$$

Adding $2y$,

$$\begin{aligned}(x-3)y' + 2y &= \sum_{n=1} n a_n x^n - \sum_{n=1} 3 n a_n x^{n-1} + \sum_{n=0} 2 a_n x^n \\ &= \sum_{n=1} n a_n x^n - \sum_{n=0} 3(n+1) a_{n+1} x^n + \sum_{n=0} 2 a_n x^n \\ &= \sum_{n=1} n a_n x^n - 3a_1 - \sum_{n=1} 3(n+1) a_{n+1} x^n + 2a_0 + \sum_{n=1} 2 a_n x^n \\ &= 2a_0 - 3a_1 + \sum_{n=1} (n a_n - 3(n+1) a_{n+1} + 2a_n) x^n \\ &= 2a_0 - 3a_1 + \sum_{n=1} ((n+2) a_n - 3(n+1) a_{n+1}) x^n.\end{aligned}$$

For this to equal zero, then $2a_0 - 3a_1 = 0$ and, for $n \geq 1$, $(n+2)a_n - 3(n+1)a_{n+1} = 0$. This means that $a_1 = \frac{2}{3}a_0$, and for $n \geq 1$, $a_{n+1} = \frac{n+2}{3(n+1)}a_n$.

Claim:[1] for $n \geq 0$, $a_n = \frac{n+1}{3^n}a_0$. We prove this by induction.

- Case $n = 1$. Then $a_0 = \frac{0+1}{3^0}a_0$.
- Case $n > 1$. Assume the claim is true for $n - 1$. Then $a_n = a_{(n-1)+1} = \frac{(n-1)+2}{3((n-1)+1)}a_{n-1} = \frac{n+1}{3n}a_{n-1} = \frac{n+1}{3n}\frac{(n-1)+1}{3^{n-1}}a_0 = \frac{n+1}{3 \cdot 3^{n-1}}a_0 = \frac{n+1}{3^n}a_0$.

This completes the induction proof, so the claim is true. Thus, $y = \sum_{n=0}^{\infty} \frac{a_0(n+1)}{3^n}x^n$.

Here is what we can do to identify this as some elementary function. Rewriting this as $y = a_0 \sum_{n=0}^{\infty}(n+1)(\frac{x}{3})^n$ and integrating, we get $\int y\,dx = 3a_0 \sum_{n=0}(\frac{x}{3})^{n+1} = \frac{3a_0}{1-\frac{x}{3}}$. So then, taking the derivative of this, we get $y = \frac{a_0}{(1-\frac{x}{3})^2} = \frac{9a_0}{(3-x)^2}$.

[1] Found by privately finding the pattern.