The sum of convergent and divergent series

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Theorem 8 in section 11.2 says (among other things) that if both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge, then so do $\sum_{n=1}^{\infty}(a_n + b_n)$ and $\sum_{n=1}^{\infty}(a_n - b_n)$.

Today I gave the example of a difference of divergent series which converges (for instance, when $a_n = b_n$), but I misspoke about what Theorem 8 says about the sum of a convergent and divergent series: the result is in fact divergent.

We will show that if the sum is convergent, and one of the summands is convergent, then the other summand *must* be convergent. Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty}(a_n + b_n)$ converge. Then $\sum_{n=1}^{\infty}(a_n + b_n) - \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} b_n$ converges, by Theorem 8. In particular, if $\sum_{n=1}^{\infty} a_n$ converges and $\sum_{n=1}^{\infty} b_n$ diverges, so must $\sum_{n=1}^{\infty}(a_n + b_n)$.

Another way of saying this is by the contrapositive: if the difference of two series diverges, then one of the two series must diverge. Use $\sum_{n=1}^{\infty}(a_n + b_n)$ and $\sum_{n=1}^{\infty} a_n$ along with the fact that $\sum_{n=1}^{\infty} a_n$ is known to converge.