11.1 Problem 91

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In this problem, we are going to show that iteratively applying arithmetic and geometric means in a certain way creates two sequences which converge to the same number. The arithmetic mean of two numbers is just half their sum, and the geometric mean is the square root of their product.[1]

Let a and b be positive numbers with $a > b$. We define two sequences a_n and b_n by having

$$a_1 = \frac{a+b}{2} \qquad\qquad b_1 = \sqrt{ab}$$

and, for $n > 1$,

$$a_{n+1} = \frac{a_n + b_n}{2} \qquad\qquad b_{n+1} = \sqrt{a_n b_n}.$$

Note that the following is *a* way to solve this problem. It is merely the way I came up with, and it is quite possible there are slicker or clearer ways of obtaining the result.

(a) We will show that $a_n > a_{n+1} > b_{n+1} > b_n$ for all $n \geq 1$.

The way the sequence is defined by Stewart is a bit annoying to work with. Because of this, we will let $a_0 = a$ and $b_0 = b$. Notice that $\frac{a_0+b_0}{2} = a_1$ and $\sqrt{a_0 b_0} = b_1$, so the rule for $n > 1$ is now a rule for $n \geq 1$.

First, the base case for the induction: we will show the inequality for $n = 0$, in particular, that $a_0 > a_1 > b_1 > b_0$. Using the fact that $a_0 > b_0$, we have $a_1 = \frac{a+b}{2} < \frac{a+a}{2} = a = a_0$, so $a_0 > a_1$. Next, since $a > b$ implies $\sqrt{a} - \sqrt{b} > 0$, we see $a_1 - b_1 = \frac{a+b}{2} - \sqrt{ab} = \frac{a - 2\sqrt{ab} + b}{2} = \frac{\left(\sqrt{a}-\sqrt{b}\right)^2}{2}$ is positive, hence $a_1 > b_1$. Finally for the base case, using $a_0 > b_0$, we have $b_1 = \sqrt{a_0 b_0} > \sqrt{b_0 b_0} = b_0$. Thus, $a_0 > a_1 > b_1 > b_0$.

Second, we prove the induction hypothesis, that $a_{n+1} > a_{n+2} > b_{n+2} > b_{n+1}$ is true whenever $a_n > a_{n+1} > b_{n+1} > b_n$ is true, for $n \geq 1$. Once we prove this, then the principal of induction lets us conclude that $a_n > a_{n+1} > b_{n+1} > b_n$ is true for all $n \geq 0$.

Thus, assume the hypothesis $a_n > a_{n+1} > b_{n+1} > b_n$. There are three inequalities to show:

(i) That $a_{n+1} > a_{n+2}$. Since $b_{n+1} < a_{n+1}$, we have $a_{n+2} = \frac{a_{n+1}+b_{n+1}}{2} < \frac{a_{n+1}+a_{n+1}}{2} = a_{n+1}$. This establishes the first inequality.

[1] Notice the parallelism: square root is the half power, and in particular, $\ln \sqrt{ab} = \frac{\ln a + \ln b}{2}$.

(ii) That $a_{n+2} > b_{n+2}$. Since $a_{n+1} > b_{n+1}$ implies $\sqrt{a_{n+1}} > \sqrt{b_{n+1}}$, the difference $a_{n+2} - b_{n+2} = \frac{a_{n+1}+b_{n+1}}{2} - \sqrt{a_{n+1}b_{n+1}} = \frac{a_{n+1}-2\sqrt{a_{n+1}b_{n+1}}+b_{n+1}}{2} = \frac{\left(\sqrt{a_{n+1}}-\sqrt{b_{n+1}2}\right)^2}{2}$ is positive, from which we obtain $a_{n+2} > b_{n+2}$.

(iii) That $b_{n+2} > b_{n+1}$. Because $a_{n+1} > b_{n+1}$, we have $b_{n+2} = \sqrt{a_{n+1}b_{n+1}} > \sqrt{b_{n+1}^2} = b_{n+1}$.

This concludes the induction.

(b) By the inequality from (a), we know that $\{a_n\}$ is a decreasing sequence, $\{b_n\}$ is an increasing sequence, that $\{a_n\}$ is bounded below by b_1, and that $\{b_n\}$ is bounded above by a_1. Thus, by Theorem 12, both sequences converge.

(c) Let $L_a = \lim_{n\to\infty} a_n$ and $L_b = \lim_{n\to\infty} b_n$. One might notice that $L_b^2 = \lim_{n\to\infty} b_{n+1}^2 = \lim_{n\to\infty} \sqrt{a_n b_n}^2 = \lim_{n\to\infty} a_n b_n = L_a L_b$. By dividing both sides by L_b, we prove $L_a = L_b$.