11.1 Problem 90

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Let $a_n = (1 + \frac{1}{n})^n$. This problem is ultimately asking us to prove that a_n converges. Is this believable? Here are some tables for parts of the sequence:

n	a_n	n	a_n	n	a_n	n	a_n
1	2.000	101	2.705	1001	2.717	10001	2.718
2	2.250	102	2.705	1002	2.717	10002	2.718
3	2.370	103	2.705	1003	2.717	10003	2.718
4	2.441	104	2.705	1004	2.717	10004	2.718
5	2.488	105	2.705	1005	2.717	10005	2.718
6	2.522	106	2.706	1006	2.717	10006	2.718
7	2.546	107	2.706	1007	2.717	10007	2.718
8	2.566	108	2.706	1008	2.717	10008	2.718
9	2.581	109	2.706	1009	2.717	10009	2.718
10	2.594	110	2.706	1010	2.717	10010	2.718

It seems to be approaching some number near 2.718, but that in itself isn't proof of convergence. It might just be increasing *extremely* slowly. For instance, try looking at $f(x) = \log_{10}(\log_{10} x)$ and check the limit as $x \to \infty$.[1]

Another question: what is the motivation for studying this function? If you are familiar with compound interest, this is the formula for the value of 1 unit of money after 1 unit of time at 100% interest with n compoundings. Generally speaking, when r is the interest rate and p is the principal, banks give $\frac{r}{n}$ interest per compounding cycle. This means that if p_i is the principal at the end of cycle i, then $p_{i+1} = (1 + \frac{r}{n})p_i$. A closed form for this is $p_i = (1 + \frac{r}{n})^i p_0$, with p_0 being the initial principal. So, this problem is $r = 1$, $p_0 = 1$, and $i = n$, with the intent to understand better what happens when the number of compounding cycles increases to ever larger numbers.

With that said, what this problem does *not* address is *what* this sequence converges *to*. It is a curious thing that mathematics can give us the knowledge of something's existence without giving us the thing itself.

(a) We will do this step in two ways, first by plain algebra, and second by making use of calculus.

[1]For instance, even $10^{10^{100}}$, which is a number with a googol digits, has $f(10^{10^{100}}) = 100$.

Let a and b be two numbers such that $0 \leq a < b$.[2] Since $b^{n+1} - a^{n+1}$ factors as $(b-a)(b^n + b^{n-1}a + b^{n-2}a^2 + \cdots + a^{n-i}a^i + \cdots + ba^{n-1} + a^n)$, we have that

$$\begin{aligned}\frac{b^{n+1} - a^{n+1}}{b-a} &= b^n + b^{n-1}a + \cdots + ba^{n-1} + a^n \\ &< b^n + b^{n-1}b + \cdots + bb^{n-1} + b^n \\ &= b^n + b^n + \cdots + b^n + b^n,\end{aligned}$$

where the inequality is from the assumption that $b > a$: everything is positive, hence $b^k a^\ell < b^k b^\ell$. Since the powers on b run from n down to 0 in the factorization, there are $n+1$ copies of b^n in the end, hence

$$\frac{b^{n+1} - a^{n+1}}{b-a} < (n+1)b^n,$$

as required.

Here is an (arguably superior) way to do it. Let $f(x) = x^{n+1}$, which is continuous and differentiable. By the mean value theorem, there is some c between a and b such that

$$\frac{b^{n+1} - a^{n+1}}{b-1} = f'(c) = (n+1)c^n.$$

And since $c < b$, we have $(n+1)c^n < (n+1)b^n$, which gives the required inequality.

For intuition as far as interpreting the mean value theorem, notice that the left side of the required result is the secant line of $f(x) = x^{n+1}$ between a and b, and that the right side is the derivative of f evaluated at b. Since f is concave up, the tangent line at the end of a secant line *must* be steeper than the secant line, which gives the inequality.

(b) With this result, we multiply both sides by the denominator to obtain

$$\begin{aligned}b^{n+1} - a^{n+1} &< (b-a)(n+1)b^n \\ &= (bn + b - a(n+1))b^n = b^{n+1} - ((n+1)a - nb)b^n.\end{aligned}$$

Subtracting b^{n+1} from both sides, we obtain

$$-a^{n+1} < -((n+1)a - nb)b^n,$$

and negating both sides, we obtain the required inequality:

$$a^{n+1} > ((n+1)a - nb)b^n.$$

(c) Let $a = 1 + \frac{1}{n+1}$ and $b = 1 + \frac{1}{n}$. First, we see that $0 \leq a$ since a is 1 plus a positive number, and second we see that $a < b$ since $n+1 > n$ implies $\frac{1}{n+1} < \frac{1}{n}$ implies

[2] Unfortunately, Stewart uses the letter a for both this number and for the sequence a_n. These are different uses of the same letter, but the subscript is enough for disambiguation.

$1 + \frac{1}{n+1} < 1 + \frac{1}{n}$, so we may use the inequality from (b). Substituting, we have

$$\begin{aligned} \left(1 + \frac{1}{n+1}\right)^{n+1} &> \left((n+1)\left(1 + \frac{1}{n+1}\right) - n\left(1 + \frac{1}{n}\right)\right)\left(1 + \frac{1}{n}\right)^{n} \\ &= ((n+2) - (n+1))\left(1 + \frac{1}{n}\right)^{n} \\ &= \left(1 + \frac{1}{n}\right)^{n}. \end{aligned}$$

Since the left side is the sequence at $n+1$ and the right side is the sequence at n, this establishes that

$$a_{n+1} > a_n,$$

and therefore, since this is true for all n, the sequence is increasing.

(d) Now let $a = 1$ and $b = 1 + \frac{1}{2n}$, and again we note $0 \leq a < b$. We substitute these into the inequality from (b) to get

$$1^{n+1} > \left((n+1) \cdot 1 - n\left(1 + \frac{1}{2n}\right)\right)\left(1 + \frac{1}{2n}\right)^{n},$$

which simplifies to

$$1 > \frac{1}{2}\left(1 + \frac{1}{2n}\right)^{n}.$$

This is equivalent to $2 > \left(1 + \frac{1}{2n}\right)^{n}$. Both sides are positive, so by squaring both sides we obtain

$$4 > \left(1 + \frac{1}{2n}\right)^{2n} = a_{2n},$$

which is the desired inequality.

(e) We now claim that a_n is bounded above by 4. We have already showed that a_n is bounded above by 4 when n is even. When n is odd, then $n+1$ is even, and we can make use of the fact a_n is an increasing sequence to say $a_n < a_{n+1} < 4$. This proves that $a_n < 4$ for all n.

(f) Therefore, since a_n is an increasing function which is bounded above, the sequence converges by Theorem 12 (i.e., $\lim_{n\to\infty} a_n$ exists).

A quick word about writing proofs: notice that in all of the arguments above, the conclusions are derived from things which are known. Never begin with the conclusion and work backwards unless you have a compelling reason to do so, and you are sure each of your steps are logically correct (of course you are free to do whatever you like in your private scratchwork). Notice also that it is written in a way which is like an essay with an audience in mind: I intended for it to be read and (hopefully) understood. "A proof is that which convinces a reasonable [person]."