Quiz 6

1. (2 points). Find two constants a and b such that the function

$$g(x) = \begin{cases} \sqrt{x} & \text{if } 0 < x < 1 \\ ax + b & \text{if } 1 \leq x \end{cases}$$

is continuous and differentiable on the domain $(0, \infty)$.

The following writeup of the solution is more detailed than you would need to give on a midterm on when calculating for your own purposes, but I think it is worth going through them here.

To be continuous, we need $\lim_{x\to c} g(x) = g(c)$ for all $c \in (0, \infty)$. When $0 < x < 1$, then $g(x) = \sqrt{x}$, so when $0 < c < 1$, $\lim_{x\to c} g(x) = \lim_{x\to c} \sqrt{x} = \sqrt{c} = g(c)$. When $1 < x$, then $g(x) = ax + b$, so when $1 < c$, $\lim_{x\to c} g(x) = \lim_{x\to c}(ax + b) = ac + b = g(c)$. So far we have shown that g is continuous on $(0, 1) \cup (1, \infty)$. What remains is when $c = 1$. If the left- and right- sided limits agree, then the limit exists:

$$\lim_{x\to 1^-} g(x) = \lim_{x\to 1^-} \sqrt{x} = \sqrt{1} = 1$$
$$\lim_{x\to 1^+} g(x) = \lim_{x\to 1^+} (ax + b) = a \cdot 1 + b = a + b$$

So, whenever $1 = a + b$, g is continuous. (There are infinitely many solutions for continuity! Think about why this makes sense.)

To be differentiable, the difference quotient limit $\lim_{h\to 0} \frac{g(x+h)-g(x)}{h}$ must exist for all $x \in (0, \infty)$. When $x \in (0, 1)$ or $x \in (1, \infty)$, then we may replace g by the corresponding $\sqrt{x}$ or $ax + b$ as before, and both of these are differentiable on their respective subdomains. We just need to check $x = 1$. Again, the left- and right- sided limits must agree for the limit to exist:

$$\lim_{h\to 0^-} \frac{g(1+h) - g(1)}{h} = \lim_{h\to 0^-} \frac{\sqrt{1+h} - 1}{h} = \frac{d}{dx}\sqrt{x}\bigg|_{x=1} = (\frac{1}{2}x^{-1/2})\bigg|_{x=1} = \frac{1}{2}$$
$$\lim_{h\to 0^+} \frac{g(1+h) - g(1)}{h} = \lim_{h\to 0^+} \frac{(ax+b) - 1}{h} = \frac{d}{dx}(ax + b)\bigg|_{x=1} = a\big|_{x=1} = a.$$

So $a = \frac{1}{2}$, and since $1 = a + b$, we can solve for b to get $b = \frac{1}{2}$.

The problem only asks that we find two constants so that the function is continuous and differentiable. We could have alternatively tried to make the derivative continuous, which would imply the derivative exists, and we would have gotten the same a that way. The solution above, though, proves that $a = b = \frac{1}{2}$ is the unique solution.

2. (3 points). (a) Find the derivative of $f(x) = \frac{x^2-1}{\sqrt{x}-1}$. Do not use the quotient or product rules. (b) What is the domain of the derivative?

The difficulty with designing this quiz was that neither the quotient nor product rules were on the homework, so the problem had to be about taking derivatives of powers and sums, but it would be too straightforward without some obfuscation, so the idea here was to show that you can sometimes find derivatives of things you did not expect you could. Unfortunately, it required seeing the trick right at the beginning.

Notice $f(x) = \frac{(x+1)(x-1)}{\sqrt{x}-1}$ and that $x - 1 = (\sqrt{x} - 1)(\sqrt{x} + 1)$ (which we did go over in section earlier in the semester). So $f(x) = (x + 1)(\sqrt{x} + 1) = x^{3/2} + x + x^{1/2} + 1$. Taking the derivative gives $f'(x) = \frac{3}{2}x^{1/2} + 1 + \frac{1}{2}x^{-1/2}$. Alternatively, one could multiply by the conjugate, $\frac{\sqrt{x}+1}{\sqrt{x}+1}$, a technique we used for limits, to get the same answer—assuming you factor $x^2 - 1$ and see the $x - 1$ in the resulting denominator.

The fact that the question is asking about the domain should alert you that the domain might require some extra thought. First, it's clear that $x > 0$ since $x^{-1/2}$ shows up in the derivative (remember, this is $\frac{1}{\sqrt{x}}$, which is not defined when $x = 0$ or $x < 0$). But, the derivative is only defined wherever the function itself is defined:

$$\lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

requires $f(x)$ to be defined, which it is not when $x < 0$ or $x = 1$. So the domain of $f'(x)$ is $(0, 1) \cup (1, \infty)$.

Since 1 is a removable discontinuity, in some sense there *is* a tangent line at that point, so $(0, \infty)$ was accepted as an answer, even if it was not technically correct.