MATH 1A — Calculus   Quiz 4   25 February 2015

You have 20 minutes to complete the quiz. No calculators.

Name:____

1. (5 points) Let $f(x) : [0,1] \to [0,1]$ be continuous. Show that there is some $x_0 \in [0,1]$ such that $f(x_0) = x_0$. We will guide you through the argument. If by chance you find the guidance more confusing than helpful, feel free to write your own proof from scratch (probably on the back so you have enough room).

First, we define $g(x) := f(x) - x$.

(a) Explain why $g(x_0) = 0$ is equivalent to $f(x_0) = x_0$. Thus, instead of solving the original problem, we will find an $x_0 \in [0,1]$ such that $g(x_0) = 0$.

Solution. $f(x_0) = x_0$ iff $0 = f(x_0) - x_0 = g(x_0)$. □

(b) Either $g(0) = 0$, $g(0) < 0$, or $g(0) > 0$. If $g(0) = 0$, we are done. Why? What is the x_0 that works?

Solution. If $g(0) = 0$, then we may take $x_0 = 0$. □

(c) In fact, it cannot be the case that $g(0) < 0$. Why? (Don't forget that f is a function into $[0,1]$, not all of $\mathbb{R}$!) Thus, unless we have found an x_0 that works in part (c), we know that it must be the case that $g(b) > 0$.

Solution. If $g(0) < 0$, then $f(0) < 0$. But we have assumed that $f(x) \geq 0$ for $x \in [0,1]$: a contradiction. Thus, we may as well assume that $g(0) > 0$. □

(d) Go through a similar argument with $g(1)$: if $g(1) = 0$ we are done; if not, what is the sign of $g(1)$?

Solution. If $g(1) = 0$, then we may take $x_0 = 1$. If $g(1) > 0$, then $f(1) > 1$: a contradiction. Thus, we may as well assume that $g(1) < 0$. □

(e) At this point, we know (i) that g is continuous[1] and (unless we have already found an x_0 in parts (b) or (d)) (ii) the sign of $g(0)$ and the sign of $g(1)$. Apply the Intermediate Value Theorem to deduce that there is some point $x_0 \in [0,1]$ such that $g(x_0) = 0$.

Solution. $g(0) > 0$ and $g(1) < 0$. Thus, by the Intermediate Value Theorem, there must be some $x_0 \in [0,1]$ such that $g(x_0) = 0$. □

[1] There are not enough points to grade you on this, but if you have time left over, you should think real quick about why we know g is continuous.