Quiz 2

1. (2 points). Let $k_0(t) = 2^t$, and let $k_1(t)$ be the function obtained by shifting the graph of k_0 by 5 units to the left. Simplify $k_1(t)/k_0(t)$. How else could one obtain $k_1(t)$ by manipulating the graph of k_0?

By applying graph rules as in Stewart, $k_1(t) = k_0(t+5)$. Notice that $k_1(-5) = k_0(-5+5) = k_0(0)$, so at $x = -5$ the graph of k_1 is the graph of k_0 at $x = 0$, so we have some evidence that we did the shift in the correct direction.

Then,

$$\frac{k_1(t)}{k_0(t)} = \frac{k_0(t+5)}{k_0(t)} = \frac{2^{t+5}}{2^t} = \frac{2^t 2^5}{2^t} = 2^5 = 32.$$

One could also obtain the graph of k_1 by scaling the graph of k_0 vertically by a factor of 32. Indeed, the above simplification gives us that $k_1(t) = 32k_0(t)$.

2. (3 points). Find the inverse f^{-1} of $f(x) = \frac{1+e^x}{1+2e^x}$. What is the domain and range of f?

To find the inverse, we will solve $y = \frac{1+e^x}{1+2e^x}$ for x. First, we multiply each side by $1+2e^x$, and then we manipulate:

$$\begin{aligned}(1+2e^x)y &= 1+e^x \\ y + 2ye^x &= 1+e^x \\ 2ye^x - e^x &= 1-y \\ (2y-1)e^x &= 1-y \\ e^x &= \frac{1-y}{2y-1} \\ x &= \ln\frac{1-y}{2y-1}.\end{aligned}$$

Hence, we have a formula for the inverse: $f^{-1}(y) = \ln\frac{1-y}{2y-1}$. One should verify that each step is completely reversible, that we did not inadvertently gain nor lose possible x's and y's, for instance by dividing or multiplying by zero in the manipulation. (And beware splitting this into two logarithms because the division law for logarithms only applies *when the numerator and denominator are both positive*.)

Let us find the domain of f. The exponential function e^x is defined for all x, so there are no considerations there. The quotient requires taht $1+2e^x \neq 0$, which is equivalent to $e^x \neq -\frac{1}{2}$. However, e^x is positive for all x, so this never happens. Therefore, the domain of f is $\mathbb{R}$.

To find the range of f, we will find the domain of f^{-1}. In this case, we have the following constrains:

$$\begin{aligned}&\frac{1-y}{2y-1} > 0 && \text{from the domain of ln} \\ &2y-1 \neq 0 && \text{from the denominator of the quotient}\end{aligned}$$

We will consider two cases:

Case I. $2y-1>0$. Then we can multiply both sides of the first inequality by $2y-1$ without flipping the $>$. This gives $1-y>0$. Together, these give $y>\frac{1}{2}$ and $y<1$, making an open interval $(\frac{1}{2},1)$.

Case II. $2y-1<0$. Then the multiplication does flip the $>$ so we get $1-y<0$. Together, these give $y<\frac{1}{2}$ and $y>1$, which is impossible.

Since only Case I is possible, the domain of f^{-1} then is just $(\frac{1}{2},1)$, and hence the range of f is $(\frac{1}{2},1)$.

Extra problems

X. (X points). Suppose the graph of a function g is given. Write the equation for the graph obtained from the graph of g after shrinking it vertically by a factor of 2, shifting it 3 units to the left, and then shifting it 4 units downward.
X. (X points). Let $f(x)=\sqrt{2-x}$. What is the domain of the function $h(x)=f(2x)/f(1-x)$? What is the domain of the inverse function h^{-1}? Find a formula for h^{-1}.