Name: ______________________________

Student ID: ______________________________

GSI: ______________________________

The exam is closed book, apart from a sheet of notes 8"x11". Calculators and smart-phones are not allowed. For full credit, you need to show all the reasoning that goes into solving the problem, step by step – the answer alone is not enough. It is your responsibility to write your answers clearly.

There are *two pages* of problems. Please write solutions in blue books.

Problem 1 ____________________

Problem 2 ____________________

Problem 3 ____________________

Problem 4 ____________________

Problem 5 ____________________

Problem 6 ____________________

Problem 7 ____________________

Total ____________________ *out of 55*

1. (8 points)

(a) (2 points) Apply the product rule:

$$f'(x) = e^x + xe^x.$$

(b) (2 points)

$$f''(x) = e^x + (e^x + xe^x) = 2e^x + xe^x.$$

(c) (4 points) The 17th derivative is

$$17e^x + xe^x.$$

2. (9 points)

(a) (3 points)

$$g'(x) = \frac{1}{2\sqrt{1 - \frac{1}{e^{3x}+1}}} \cdot \frac{1}{(e^{3x}+1)^2} \cdot 3e^{3x}$$

(b) (4 points)

$$y - g(0) = g'(0)(x - 0)$$
$$y - \frac{1}{\sqrt{2}} = \frac{3}{4\sqrt{2}}x$$
$$y = \frac{3}{4\sqrt{2}}x + \frac{1}{\sqrt{2}} \tag{1}$$

(c) (2 points) We plug in 0.1 for the x value in (1):

$$y = \frac{3}{4\sqrt{2}}(0.1) + \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}(\frac{43}{40}) \approx \frac{1.07}{1.41} \approx 0.8.$$

3. (7 points) *Taking logarithms,*

$$g(x) = (x^2+1)^{(x^2+1)} \implies \ln g(x) = (x^2+1)\ln(x^2+1)$$

Differentiating,

$$\frac{g'(x)}{g(x)} = 2x\ln(x^2+1) + (x^2+1)\frac{1}{x^2+1} \cdot 2x = 2x(1 + \ln(x^2+1)).$$

Substituting for $g(x)$ and simplifying,

$$g'(x) = 2x\left(x^2+1\right)^{x^2+1}\left(\ln\left(x^2+1\right)+1\right).$$

4. (8 points) *Differentiating implicitly,*

$$2x + 8x^3 + 6yy' = 0.$$

For the tangent to be horizontal, we must have $y' = 0$. *Therefore at such points,*

$$2x + 8x^3 = 0 \implies x = 0.$$

Substituting $x = 0$ *into the defining equation* $x^2 + 2x^4 + 3y^2 = 5$,

$$3y^2 = 5 \implies y = \pm\sqrt{\frac{5}{3}}.$$

Therefore the desired points are

$$(0, \sqrt{\frac{5}{3}}), (0, -\sqrt{\frac{5}{3}}).$$

5. (6 points) *For the function to be second-differentiable, we must have*

$$(3t^2)'' = (at^2 + bt + c)'' \text{ at } t = 1.$$

Therefore

$$6 = 2a \implies a = 3.$$

Similarly, for differentiability, we must have

$$(3t^2)' = (3t^2 + bt + c)' \text{ at } t = 1.$$

Therefore

$$6 = 6 + b \implies b = 0.$$

Finally, continuity implies that

$$3t^2|_{t=1} = (3t^2 + c)|_{t=1}$$

$$3 = 3 + c \implies c = 0.$$

In summary,

$$(a, b, c) = (3, 0, 0).$$

6. (8 points) *Let*

$$f(x) = \frac{x}{x^2 + 4}$$

(a) (2 points) Differentiating,

$$f'(x) = \frac{4 - x^2}{(x^2 + 4)^2}$$

Therefore the critical points are $x = \pm 2$.

(b) (4 points) At the end points,

$$f(10) = \frac{5}{52}$$
$$f(-10) = -\frac{5}{52}.$$

At the critical points,

$$f(2) = \frac{1}{4}$$
$$f(-2) = -\frac{1}{4}.$$

Consequently, the absolute maximum and minimum are $\frac{1}{4}$ and $-\frac{1}{4}$.

(c) (2 points) Local extrema of differentiable functions are critical points. In part (a), we saw that $f'(x) = 0 \iff x = \pm 2$.

7. (9 points)

(a) (3 points) By definition,

$$f(f^{-1}(x)) = x.$$

Differentiating both sides and using the chain rule,

$$f'(f^{-1}(x)) \cdot (f^{-1})'(x) = 1.$$

Therefore

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}.$$

(b) (2 points) Using the formula from part (a),

$$(f^{-1})'(4) = \frac{1}{f'(f^{-1}(4))} = \frac{1}{f'(3)} = \frac{3}{2}.$$

(c) (4 points) Recall that $[\csc(x)]' = -\csc(x)\cot(x)$. By part (a),

$$[arc \csc x]' = \frac{1}{-\csc(arc \csc x)\cot(arc \csc x)}.$$

It is immediate that $\csc(arc \csc x) = x$. To determine $\cot(arc \csc x)$, we use the trigonometric identity

$$1 + \cot^2(\theta) = \csc^2(\theta).$$

Set $\theta = arc\csc x$*. Then*

$$1 + \cot^2(arc\csc x) = \csc^2(arc\csc x) = x^2,$$

so that

$$\cot(arc\csc x) = \sqrt{x^2 - 1}.$$

We have shown that

$$[arc\csc x]' = \frac{1}{-x\sqrt{x^2 - 1}}.$$