Name: ______________________________

Student ID: ______________________________

GSI: ______________________________

The exam is closed book, apart from a sheet of notes 8”x11”. Calculators and smart-phones are not allowed. For full credit, you need to show all the reasoning that goes into solving the problem, step by step – the answer alone is not enough. It is your responsibility to write your answers clearly.

There are *two pages* of problems. Please write solutions in blue books.

Problem 1 ____________________

Problem 2 ____________________

Problem 3 ____________________

Problem 4 ____________________

Problem 5 ____________________

Problem 6 ____________________

Total ____________________ *out of 30*

1. (4 points) *Let*

$$f(x) = \frac{(x^2+1)(x-1)^2}{(x^2-1)^2}$$

Find the following limits (possibly as infinite limits). Justify your answers.

(a)

$$\lim_{x\to 0} f(x) = f(0) = 1$$

(b)

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{(x^2+1)}{(x+1)^2} = \frac{1}{2}$$

(c)

$$\lim_{x\to -1} f(x) = \lim_{x\to -1} \frac{(x^2+1)}{(x+1)^2} = \infty$$

Since the numerator tends to 2 and the denominator is positive and tends to 0.

(d) Is $f(x)$ continuous? Explain your answer.

$f(x)$ isn't continuous as it has a removable discontinuity at $x = 1$ and an infinite discontinuity at $x = -1$.

2. (6 points) *Let*

$$f(x) = 1 + \sqrt{x-1}, \qquad g(x) = \ln(2x-1).$$

(a) Find the domain of the function $f(x)$.

The domain is $[1, \infty)$, since we need $x - 1 \geq 0$.

(b) Find the domain of the function $g(x)$.

The domain is $(\frac{1}{2}, \infty)$ since we need $2x - 1 > 0$, which means $2x > 1$, so that $x > \frac{1}{2}$.

(c) Find $g \circ f(x)$ and its domain.

$g \circ f(x) = \ln(2(1+\sqrt{x-1}) - 1) = \ln(1 + 2\sqrt{x-1})$. The domain is $[1, \infty)$.

(d) Find $f \circ g(x)$ and its domain.

$f \circ g(x) = 1 + \sqrt{\ln(2x-1) - 1}$. To find the domain, we need to solve:

$$\begin{aligned} \ln(2x-1) - 1 &\geq 0 \\ 2x - 1 &\geq e \\ x &\geq \frac{e+1}{2} \end{aligned}$$

The domain is $[\frac{e+1}{2}, \infty)$.

3. (5 points) *Let*

$$f(x) = \sqrt{1 - \frac{1}{e^x + 1}}.$$

Find the inverse of $f(x)$. Is 7 in the range of $f(x)$?

First we switch x and y and solve for y.

$$\begin{aligned} x &= \sqrt{1 - \frac{1}{e^y + 1}} \\ 1 - x^2 &= \frac{1}{e^y + 1} \\ \frac{1}{1 - x^2} - 1 &= e^y \\ y &= \ln\left(\frac{1}{1 - x^2} - 1\right) \end{aligned}$$

If we try to solve $f(x) = 7$, we get $x = \ln(\frac{1}{1-49} - 1)$ which is undefined since $\frac{1}{1-49} - 1 < 0$.

4. (5 points) *Let*

$$f(x) = x^4 \sin\left(\frac{5}{x}\right).$$

Use the squeeze theorem to find

$$\lim_{x \to 0} f(x).$$

Since $-1 \leq \sin(\frac{5}{x}) \leq 1$, we have $-x^4 \leq x^4 \sin(\frac{5}{x}) \leq x^4$. Since

$$\lim_{x \to 0} \pm x^4 = 0$$

we have

$$\lim_{x \to 0} f(x) = 0.$$

5. (5 points) *Let*

$$h(t) = \begin{cases} a, & t < 2 \\ 1 + at, & 2 \leq t \end{cases}$$

(a) For which value of the constant a is the function $h(t)$ continuous for all t?

Clearly we only have to worry about the function being continuous at $t = 2$. We compute the limit on both sides:

$$\lim_{t \to 2^-} h(t) = a$$

From the right

$$\lim_{t \to 2^+} h(t) = 1 + 2a$$

If we want these to be equal, we have $a = 1 + 2a$, so $a = -1$.

(b) Write down the function $g(t)$ related to $h(t)$ by shifting the graph of $h(t)$ one unit to the left.

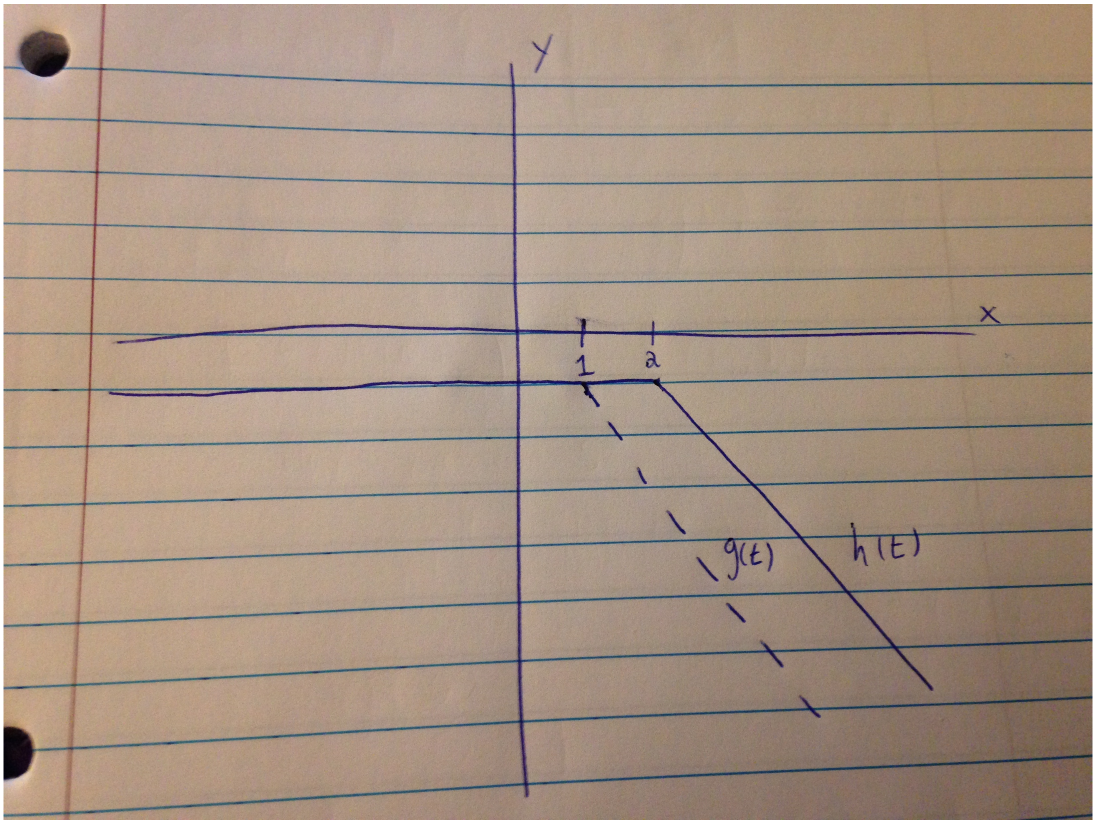
To shift the graph, we need to replace t with $t + 1$. That is:

$$g(t) = \begin{cases} a, & t + 1 < 2 \\ 1 + a(t + 1), & 2 \leq t + 1 \end{cases}$$

Simplifying and letting $a = -1$

$$g(t) = \begin{cases} -1, & t < 1 \\ -t, & 1 \leq t \end{cases}$$

(c) Having found a, sketch the graph of $h(t)$ and $g(t)$. (Your sketch should be at least qualitatively correct.)



6. (5 points) *Recall the definition of*

$$\lim_{x \to 2}(2x + 3) = 7$$

(a) For every $\epsilon > 0$ there exists $\delta > 0$ such that

$$\text{if} \qquad |x - A| < B, \qquad \text{then} \qquad |2x + 3 - C| < D$$

Find the missing numbers A, B, C and D in the definition:

$A = 2$
$B = \delta$
$C = 7$
$D = \epsilon$

(b) For a given $\epsilon > 0$, find a $\delta > 0$ for which the statement holds. If $\epsilon = 1/10$, what is δ?

Since $2x+3-7 = 2(x-2)$, if $|x-2| < \delta$, we should let $\delta = \frac{\epsilon}{2}$ so that $|2x+3-7| < 2\delta = \epsilon$.