Quiz 8

1. (5 points). Compute the derivative of $H(z) = \ln\sqrt{\frac{a^2-z^2}{a^2+z^2}}$.

By using rules for logarithms, we may rewrite H as

$$\begin{aligned} H(z) &= \ln\sqrt{\frac{a^2-z^2}{a^2+z^2}} \\ &= \frac{1}{2}\ln\left(\frac{a^2-z^2}{a^2+z^2}\right) \\ &= \frac{1}{2}\left(\ln(a^2-z^2) - \ln(a^2+z^2)\right). \end{aligned}$$

With this, it becomes easy to compute the derivative:

$$\begin{aligned} H'(z) &= \frac{1}{2}\left(\frac{-2z}{a^2-z^2} - \frac{2z}{a^2+z^2}\right) \\ &= \frac{-z(a^2+z^2) - z(a^2-z^2)}{a^4-z^4} \\ &= \frac{-2a^2z}{a^4-z^4}. \end{aligned}$$

Note that H is a function of z and z only (which is why it is written $H(z)$). That is, a is a constant.

It is also possible to compute the derivative directly using the chain rule, however it takes more work to simplify the result.

2. (5 points). Compute the derivative of $f(x) = (x^2-1)^{\sin x}$.

This function has an exponential with a base and exponent which are both varying with x, so our normal methods do not apply. There are two ways to compute this derivative (both substantially the same). The first is by logarithmic differentiation. Write $y = (x^2-1)^{\sin x}$ and take the logarithm of both sides to obtain $\ln y = \sin(x)\ln(x^2-1)$. Then, we implicitly differentiate each side,

$$\frac{1}{y}\frac{dy}{dx} = \sin(x)\frac{2x}{x^2-1} + \cos(x)\ln(x^2-1),$$

and multiply by y to solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \left(\frac{2x\sin(x)}{x^2-1} + \cos(x)\ln(x^2-1)\right)\sin(x)\ln(x^2-1).$$

Alternatively, we may realize that $(x^2-1)^{\sin x} = e^{\sin(x)\ln(x^2-1)}$ and proceed by using the chain rule followed by the product rule.

3. (5 points). Boyle's Law for an ideal gas at constant temperature with pressure P and volume V is $PV = C$, where C is some positive constant. Our gas has $C = 104$ kPa $\cdot$ m^3. Suppose at $t = 10$ s, $V = 2$ m^3, $P = 52$ kPa, and $\frac{dV}{dt} = -\frac{1}{2}$ m^3/s. What is $\frac{dP}{dt}$ at this time?

We want to find $\frac{dP}{dt}$, so therefore we must take the derivative of some relation involving P. The only relation we have is $PV = C$, so let us take the derivative with respect to t, recognizing that C is a constant:

$$P\frac{dV}{dt} + \frac{dP}{dt}V = 0.$$

Then we may substitute values we know:

$$52 \cdot \left(-\frac{1}{2}\right) + \frac{dP}{dt} \cdot 2 = 0,$$

and therefore $\frac{dP}{dt} = \frac{52}{2 \cdot 2} = 13$ kPa/s.

Extra credit. (2 points). Suppose $y(x)$ is a function satisfying $xy'' + y' + xy = 0$ for all values x, and $y(0) = 1$. Find $y'(0)$ and $y''(0)$. (This y is called a Bessel function of order 0).

Substitute $x = 0$, and then we have $0 \cdot y''(0) + y'(0) + 0 \cdot y(0) = 0$, and so $y'(0) = 0$. Take the implicit derivative with respect to x to get $(xy''' + y'') + y'' + (xy' + y) = 0$, into which we substitute $x = 0$ to get $0 \cdot y'''(0) + 2y''(0) + 0 \cdot y'(0) + y(0) = 0$, which means $2y''(0) + 1 = 0$, and therefore $y''(0) = -\frac{1}{2}$.