Quiz 7

1. (5 points). Find the derivative of $f(t) = e^{t \sin 2t}$.

We compute the derivative as follows:

$$\begin{aligned}
\frac{d}{dt} f(t) &= \frac{d}{dt} e^{t \sin 2t} \\
&= e^{t \sin 2t} \left(\frac{d}{dt} t \sin 2t \right) && \text{(chain rule)} \\
&= e^{t \sin 2t} \left(t \left(\frac{d}{dt} \sin 2t \right) + \left(\frac{d}{dt} t \right) \sin 2t \right) && \text{(product rule)} \\
&= e^{t \sin 2t} \left(t \cos(2t) \left(\frac{d}{dt} 2t \right) + 1 \cdot \sin 2t \right) && \text{(chain rule)} \\
&= e^{t \sin 2t} \left(t \cos(2t) \cdot 2 + \sin 2t \right).
\end{aligned}$$

Thus, the derivative is $(2t \cos 2t + \sin 2t)\, e^{t \sin 2t}$.

2. (5 points). For the curve $x^2 + xy + y^2 = 7$, find an equation of the tangent line at the point $(2, 1)$.

We compute the implicit derivative so that we have the slope at that point by taking d of each side:

$$\begin{aligned}
d(x^2 + xy + y^2) &= d(7) \\
d(x^2) + d(xy) + d(y^2) &= 0 \\
2x\, dx + x\, dy + y\, dx + 2y\, dy &= 0 \\
(2x + y)dx + (x + 2y)dy &= 0.
\end{aligned}$$

Putting in $(x, y) = (2, 1)$, we have

$$5\, dx + 4\, dy = 0$$

hence $\frac{dy}{dx} = -\frac{5}{4}$. Using point-slope form, we have the following equation for the tangent line:

$$y - 1 = -\frac{5}{4}(x - 2).$$

3. (5 points). Estimate $\sqrt[3]{29}$ by a linear approximation (i.e., by using a tangent line at 27).

In this solution, we will go over the reasoning behind linear approximation; it can be solved much more quickly in practice.

The derivative of $\sqrt[3]{x} = x^{1/3}$ is $\frac{1}{3x^{2/3}}$. Recall that the definition of the derivative is:

$$\lim_{x \to 27} \frac{\sqrt[3]{x} - \sqrt[3]{27}}{x - 27} = \frac{1}{3(27)^{2/3}} = \frac{1}{27}.$$

For every $\varepsilon > 0$, there is a $\delta > 0$ such that any x within δ of 27 makes the expression inside the limit within ε of $1/27$. So, if ε is small enough, we can remove the limit and replace $=$ with $\approx$, for x within δ of 27:

$$\frac{\sqrt[3]{x} - \sqrt[3]{27}}{x - 27} \approx \frac{1}{27}.$$

Multiplying by $x - 27$, we essentially obtain a tangent line:

$$\sqrt[3]{x} - \sqrt[3]{27} \approx \frac{1}{27}(x - 27).$$

(In particular, the tangent line at 27 is $y - 3 = \frac{1}{27}(x - 27)$.)

We would like an approximation of $\sqrt[3]{29}$, so let $x = 29$. Then, we obtain $\sqrt[3]{29} \approx \sqrt[3]{27} + \frac{1}{27}(29 - 27) = 3 + \frac{2}{27}$ as the approximation, which correct to two parts in a thousand.

Why did we do the tangent line at 27? It is because we can easily compute $\sqrt[3]{27}$, and it is the closest cube to 29.

Extra credit. (2 points). A tangent line of $h(x) = \frac{1}{x}$ meets the x- and y-axes to form a triangle. Find the area of such triangles.

Since $h'(x) = -\frac{1}{x^2}$, the tangent line at $(a, \frac{1}{a})$ is

$$y - \frac{1}{a} = -\frac{1}{a^2}(x - a).$$

By some manipulation, we obtain $\frac{x}{a} + ay = 2$. This form makes it easy to compute the x-intercept as $2a$ and the y-intercept as $\frac{2}{a}$. The area of the triangle bounded by the axes and the tangent line is $\frac{1}{2}(2a)(\frac{2}{a}) = 2$. Always 2.