Quiz 6

1. (8 points). Find a, b, c so that the following function f is differentiable everywhere and so that the derivative f' is differentiable everywhere. Recall that the derivative of e^x is e^x.

$$f(x) = \begin{cases} e^x & \text{if } x < 0 \\ ax^2 + bx + c & \text{if } 0 \leq x \end{cases}$$

For the derivative to exist, the function must be continuous. The function is continuous for all $x < 0$ and $x > 0$, so what remains to be seen is $x = 0$. For it to be continuous here, we need $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x)$, that is, $e^0 = a \cdot 0^2 + b \cdot 0 + c$, so $c = 1$.

The derivative of f exists everywhere that $x \neq 0$ since $\frac{d}{dx} e^x = e^x$ and $\frac{d}{dx}(ax^2 + bx + c) = 2ax + b$ are both defined everywhere. For the derivative to exist at $x = 0$, the left- and right-derivatives must agree, in particular e^x must equal $2ax + b$ at $x = 0$, so $b = 1$.

And, for the derivative of the derivative to exist, we check the same thing. The derivatives $\frac{d^2}{dx^2} e^x = e^x$ and $\frac{d^2}{dx}(ax^2 + bx + c) = 2a$ are defined everywhere, and they must agree at $x = 0$, so $a = \frac{1}{2}$. (We computed that the first three terms of the Taylor series of e^x at 0 are $1 + x + \frac{1}{2}x^2$. We could also do the exercise of finding a degree-n polynomial, where every derivative including the nth derivative agrees with e^x at 0.)

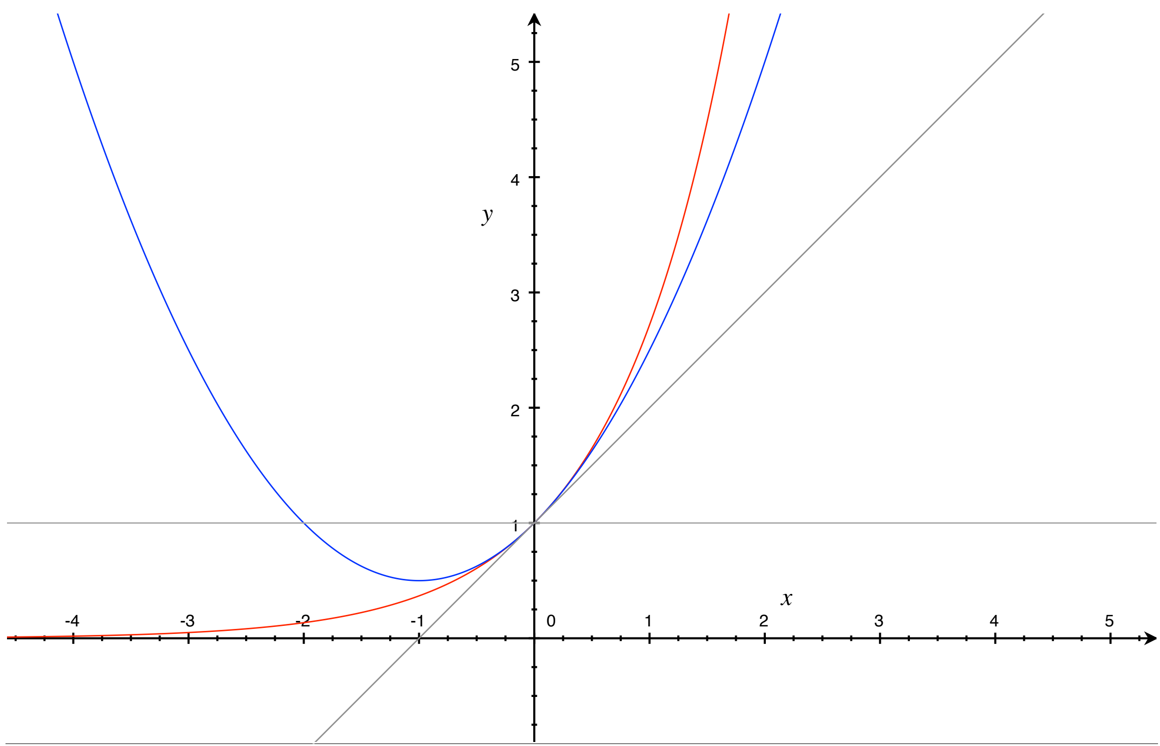

Figure 1: Red is e^x, blue is $1 + x + \frac{1}{2}x^2$. In gray are 1 and $1 + x$.

2. (8 points). Compute the derivative of $f(x) = \frac{1}{1+x^2}$, and find an equation of the tangent

line through the point $(-1, \frac{1}{2})$.

We could either do the limit definition of the derivative, use the quotient rule, or use the reciprocal rule. We will compute the derivative using the reciprocal rule here: $f'(x) = \frac{-(1+x^2)'}{(1+x^2)} = \frac{-2x}{1+x^2}$.

The derivative at -1 is $f'(-1) = \frac{-2(-1)}{1+(-1)^2} = \frac{2}{4} = \frac{1}{2}$. Thus, we obtain the following equation for the tangent line:

$$y - \frac{1}{2} = \frac{1}{2}(x - (-1))$$

which may also be written as $y = \frac{1}{2}x + 1$.

Extra credit. (2 points). Let θ be the measure of an angle centered in a unit circle. The angle subtends an arc and a chord, measured by a and c, respectively. Find $\lim_{\theta\to 0^+} \frac{a}{c}$.

Bisect the angle with a line, and this line bisects the chord, giving us the relation

$$c = 2\sin\left(\frac{1}{2}\theta\right),$$

and the length of an arc on a unit circle is just the angle of the subtending angle, so $a = \theta$. Thus, we must compute

$$\lim_{\theta\to 0^+} \frac{a}{c} = \lim_{\theta\to 0^+} \frac{\theta}{2\sin\left(\frac{1}{2}\theta\right)}.$$

By l'Hopital's rule, since the numerator and denominator each approach 0, this is $\lim_{\theta\to 0^+} \frac{1}{\cos\left(\frac{1}{2}\theta\right)}$, and since $\cos\left(\frac{1}{2}\theta\right) \to 1$ as $\theta \to 0$, we have that the limit approaches 1.

Instead of l'Hopital's rule, this limit can also be found by an application of the squeeze theorem, as is done when we computed the limit of $\frac{\sin\theta}{\theta}$.