Quiz 5

1. (5 points). Compute the limit $\lim_{x\to\infty} \dfrac{x^2(1+\cos x)}{x^3+2}$ using the squeeze theorem.

Because of the domain of cosine, $-1 \le \cos x \le 1$, and so $0 \le 1+\cos x \le 2$. Multiplying all of this by $\frac{x^2}{x^3+2}$, we have $0 \le \frac{x^2(1+\cos x)}{x^3+2} \le \frac{2x^2}{x^3+2}$.

The limit of the left side is $\lim_{x\to\infty} 0 = 0$, and the right side is $\lim_{x\to\infty} \frac{2x^2}{x^3+2} = 0$ since the degree of the polynomial on the bottom is greater than the degree of the polynomial on the top. By the squeeze theorem, the limit of the middle is 0.

2. (5 points). Compute the derivative of $f(x) = x^2 + 2$ at a using $\lim_{h\to 0} \dfrac{f(a+h)-f(a)}{h}$.

We just substitute f:

$$\begin{aligned}
\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} &= \lim_{h\to 0} \frac{((a+h)^2+2)-(a^2+2)}{h} \\
&= \lim_{h\to 0} \frac{a^2+2ah+h^2-a^2}{h} \\
&= \lim_{h\to 0} (2a+h) \\
&= 2a.
\end{aligned}$$

3. (5 points). Prove $\lim_{x\to 2}(x^2+2) = 6$ using the precise definition of the limit.

There are multiple ways of doing this, and, like always, multiple possible values of δ. Here are a couple.

One way: Let $\varepsilon > 0$, and let $\delta = \min\{1, \frac{\varepsilon}{5}\}$. Then, when x satisfies $0 < |x-2| < \delta$, we have $|(x^2+2)-6| = |x^2-4| = |x-2||x+2|$. Now, since $|x-2| < 1$, $|x+2| = |x-2+4| \le |x-2|+4 < 1+4 = 5$, hence, $|x-2||x+2| < \frac{\varepsilon}{5}\cdot 5 = \varepsilon$. Therefore, we have proved the limit.

Another way (derived from a graphical method): Let $\varepsilon > 0$, and let $\delta = \min\{2 - \sqrt{4-\varepsilon}, \sqrt{4+\varepsilon}-2\}$. For positive x, the function $f(x) = x^2+6$ is increasing, so $f(\sqrt{4-\varepsilon}) < f(x) < f(\sqrt{4+\varepsilon})$ when $\sqrt{4-\varepsilon} < x < \sqrt{4+\varepsilon}$ (which is true whenever x satisfies $0 < |x-2| < \delta$ by a little algebra). This inequality becomes $6-\varepsilon < x^2+2 < 6+\varepsilon$, that is, $|(x^2+2)-6| < \varepsilon$.

Extra credit. (2 points). With the domain $[-\frac{1}{2}, \infty)$, $f(x) = \sqrt{1+x+x^2}$ is invertible. Find a formula for the inverse f^{-1}, as well as the domain and the range of the inverse.

If we complete the square, $f(x) = \sqrt{(x+\frac{1}{2})^2+\frac{3}{4}}$, which makes it fairly easy to find some plausible inverses. Solving for x, we have

$$x = -\frac{1}{2} \pm \sqrt{f(x)^2 - \frac{3}{4}}$$

We need to get rid of the $\pm$, but one can verify $+$ is the appropriate one by examining the domain of f. Thus,

$$f^{-1}(x) = -\frac{1}{2} + \sqrt{f(x)^2 - \frac{3}{4}}$$

Using the completed square, notice that f reaches its minimum value when $x = -\frac{1}{2}$, so the range of f is $[\frac{\sqrt{3}}{2}, \infty)$. This means the domain of f^{-1} is the range of f, $[\frac{\sqrt{3}}{2}, \infty)$, and the range of f^{-1} is the domain of f, $[-\frac{1}{2}, \infty)$.

Rather than completing the square, the inverse can be found by squaring both sides and then applying the quadratic formula. This is where the quadratic formula comes from anyway.