Quiz 4

1. (5 points). Compute the limit $\lim_{x \to 0}(3e^x + 2(x+3))$. You may assume the exponential function is continuous everywhere.

Since e^x is continuous, $\lim_{x\to 0} e^x = e^0 = 1$. So, by the product and addition rules, $\lim_{x\to 0}(3e^x + 2(x+3)) = 3e^0 + 2(0+3) = 3 \cdot 1 + 6 = 9$.

2. (5 points). Find the largest δ such that, whenever x satisfies $0 < |x-2| < \delta$, $|\frac{1}{x^3} - \frac{1}{2^3}| < 1$.

The theory underlying this problem is that $\lim_{x\to 2} \frac{1}{x^3} = \frac{1}{2^3}$. We are given $\varepsilon = 1$, and it is our job to find the largest δ, where the inequalities come from the definition of the limit.

It is easiest to graph $y = \frac{1}{x^3}$ and follow the techniques we have done in class. Instead, we will do it entirely algebraically here. The second inequality is equivalent to the following two inequalities:

$$\frac{1}{x^3} - \frac{1}{2^3} < 1 \qquad\qquad \frac{1}{x^3} - \frac{1}{2^3} > -1,$$

which can be simplified to

$$\frac{1}{x^3} < \frac{9}{8} \qquad\qquad \frac{1}{x^3} > -\frac{7}{8}$$

$$x > \sqrt[3]{\frac{9}{8}} \qquad\qquad x < -\sqrt[3]{\frac{7}{8}}$$

(the inequalities flip because we took the reciprocal). The second inequality is irrelevant because $\frac{1}{x^3}$ is undefined at $x = 0$, and we are only concerned with values of $\frac{1}{x^3}$ near $x = 2$. Since we are left only with the inequality $x > \sqrt[3]{\frac{9}{8}}$, so $2 - x < 2 - \sqrt[3]{\frac{9}{8}}$, which, since both $2 - x$ and $2 - \sqrt[3]{\frac{9}{8}}$ are positive, gives $|x-2| < 2 - \sqrt[3]{\frac{9}{8}}$ (using $|x-2| = |2-x|$). Hence, we may select $\delta = 2 - \sqrt[3]{\frac{9}{8}}$, which is the largest possible δ by construction.

The drawing-a-graph method, though, is probably less error-prone than doing it purely algebraically.

3. (5 points). Let m and b be real numbers such that $m \neq 0$. Prove that $\lim_{x\to a}(mx+b)$ exists for all a, using the definition of the limit.

First, using what we know about limits, we would expect $\lim_{x\to a}(mx+b) = ma + b$. We need to prove this. Importantly, we must recall what it means for this limit to exist with this value: For all $\varepsilon > 0$, there exists some $\delta > 0$ such that whenever any x satisfies $0 < |x-a| < \delta$, we have $|(mx+b) - (ma+b)| < \varepsilon$. Let us derive a δ before actually writing the proof. Notice the conclusion can be simplified to $|m||x-a| < \varepsilon$, so if $|x-a| < \frac{\varepsilon}{|m|}$, we are good, so we will use this for our δ. Now, this was all derivation; we must write a proof:

Let $\varepsilon > 0$, and let $\delta = \frac{\varepsilon}{|m|}$. Suppose x satisfies $0 < |x-a| < \delta$. Then, $|(mx+b)-(ma+b)| = |m||x-a| < |m| \cdot \frac{\varepsilon}{|m|} = \varepsilon$. Since ε was arbitrary, we conclude $\lim_{x\to a}(mx+b) = ma+b$. $\square$

Extra credit. (2 points). Prove from the definition of the limit that $f(x) = \sqrt{x}$ is continuous at all $x > 0$.

For this, we observe $|\sqrt{x} - \sqrt{a}||\sqrt{x} + \sqrt{a}| = |x - a|$.

Let $\varepsilon > 0$, and let $\delta = \varepsilon\sqrt{a}$. Suppose x satisfies $0 < |x - a| < \delta$. Then

$$\begin{aligned} |\sqrt{x} - \sqrt{a}| &= \frac{|x-a|}{\sqrt{x} + \sqrt{a}} \\ &\leq \frac{|x-a|}{\sqrt{a}} \text{ (since } \sqrt{x} + \sqrt{a} \geq \sqrt{a}) \\ &< \frac{\varepsilon\sqrt{a}}{\sqrt{a}} \\ &= \varepsilon. \end{aligned}$$

Therefore, since ε was arbitrary, we see $\lim_{x\to a} \sqrt{x} = \sqrt{a}$, and since this argument works for all $a > 0$, $\sqrt{x}$ is continuous on the interval $(0, \infty)$. □

Note that the argument breaks down when $a = 0$: $\delta = \varepsilon\sqrt{0} = 0$, but the definition of the limit requires $\delta > 0$. Also, $\sqrt{x}$ is not defined for negative numbers, so a limit *cannot* exist at 0 anyway (though the right-sided limit does, so by Stewart, we say $\sqrt{x}$ is continuous on the interval $[0, \infty)$).