Quiz 3

1. (5 points). Simplify $\sin(2\sec^{-1}(x))$ so that it no longer has trigonometric functions.

First we note that $\sin(2\theta) = 2\sin\theta\cos\theta$, so for $\theta = \sec^{-1}(x)$, we have $\sin(2\sec^{-1}(x)) = 2\sin(\sec^{-1}(x))\cos(\sec^{-1}(x))$.

We may proceed by drawing a triangle with angle θ and for which $\sec x = \theta$, i.e., with x on the hypotenuse and 1 on the adjacent side (because the secant is the reciprocal of the cosine). The opposite side has length $\sqrt{x^2-1}$, so $\sin\theta = \frac{\sqrt{x^2-1}}{x}$ and $\cos\theta = \frac{1}{x}$. These may be combined to obtain the simplified expression $\frac{2\sqrt{x^2-1}}{x^2}$.

Alternatively, we observe $\theta = \sec^{-1}(x)$ means $x = \sec\theta = \frac{1}{\cos\theta}$. So, taking the reciprocal of both sides, we have $\frac{1}{x} = \cos\theta$, and so $\theta = \cos^{-1}\frac{1}{x}$. The expression becomes $2\sin(\cos^{-1}\frac{1}{x})\cos(\cos^{-1}\frac{1}{x})$. Since $\sin^2\theta + \cos^2\theta = 1$, we can show $\sin\theta = \sqrt{1-\cos^2\theta}$, so $\sin(\cos^{-1}\frac{1}{x}) = \sqrt{1-(\cos(\cos^{-1}\frac{1}{x}))^2} = \sqrt{1-\frac{1}{x^2}}$. Hence, the simplified expression is $2\left(\sqrt{1-\frac{1}{x^2}}\right)\left(\frac{1}{x}\right) = \frac{2\sqrt{x^2-1}}{x^2}$.

2. We define the function f by the following:

$$f(x) = \begin{cases} x & \text{if } x < 2 \\ x^2 + a & \text{if } x > 2. \end{cases}$$

(a) (2 points). What is $\lim_{x\to 2^-} f(x)$?

The left-sided limit $x \to 2^-$ means $x < 2$, so $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} x = 2$.

(b) (2 points). What is $\lim_{x\to 2^+} f(x)$?

The right-sided limit $x \to 2^+$ means $x > 2$, so $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} x^2 + a = 2^2 + a = 4 + a$.

(c) (2 points). For which a does $\lim_{x\to 2} f(x)$ exist?

A limit only exists when its left- and right-sided limits exist and are equal. The limits exist, and they equal when $a = -2$ since $2 = 4 + (-2)$.

3. (5 points). Find the limit $\lim_{x\to-\infty} \frac{2x^2-3x+1}{(3x+1)^2}$, if it exists.

Since this is quadratic divided by quadratic, the limit should exist. Let us divide the top and bottom by x^2, which we can do because we may assume x is sufficiently far from 0:

$$\begin{aligned}\lim_{x\to-\infty} \frac{2x^2-3x+1}{(3x+1)^2} \cdot \frac{1/x^2}{1/x^2} &= \lim_{x\to-\infty} \frac{2x^2-3x+1}{9x^2+6x+1} \cdot \frac{1/x^2}{1/x^2} \\ &= \lim_{x\to-\infty} \frac{2 - \frac{3}{x} + \frac{1}{x^2}}{9 + \frac{6}{x} + \frac{1}{x^2}}\end{aligned}$$

Since, for example, $\lim_{x\to-\infty} \frac{1}{x^2} = 0$, the limit of the top of the expression is 2. Likewise, the limit of the bottom of the expression is 9. Hence, the limit itself is $\frac{2}{9}$.