Quiz 2

1. (5 points). Solve the equation $5^{2z-1} = 3^{5z-1}$ for z. Express the solution with as few logarithms as possible.

   Take a logarithm of both sides (I will use the natural logarithm, but any base will do) to get

$$\begin{aligned} \ln 5^{2z-1} &= \ln 3^{5z-1} \\ (2z-1)\ln 5 &= (5z-1)\ln 3 \\ 2z\ln 5 - \ln 5 &= 5z\ln 3 - \ln 3 \\ 2z\ln 5 - 5z\ln 3 &= \ln 5 - \ln 3 \\ (2\ln 5 - 5\ln 3)z &= \ln 5 - \ln 3 \\ z &= \frac{\ln 5 - \ln 3}{2\ln 5 - 5\ln 3}. \end{aligned}$$

   Using the facts like $\ln 5 - \ln 3 = \ln \frac{5}{3}$ and $2\ln 5 = \ln 5^2$, we can rewrite this as

$$z = \frac{\ln \frac{5}{3}}{\ln \frac{5^2}{3^5}},$$

   and, using the base change formula for logarithms,

$$z = \log_{5^2/3^5} \frac{5}{3}.$$

   This is a single logarithm, so we can't reduce the number of logarithms any further.

2. (5 points). Find a formula for the inverse of the function $f(x) = \frac{2x+1}{3x-4}$. What are the domains of both f and its inverse?

   To invert this function, we can solve for x:

$$\begin{aligned} f(x) &= \frac{2x+1}{3x-4} \\ (3x-4)f(x) &= 2x+1 \\ 3xf(x) - 4f(x) &= 2x+1 \\ 3xf(x) - 2x &= 4f(x)+1 \\ (3f(x)-2)x &= 4f(x)+1 \\ x &= \frac{4f(x)+1}{3f(x)-2} \end{aligned}$$

   Using the substitution $y = f(x)$, we have $f^{-1}(y) = x$, and so

$$f^{-1}(y) = \frac{4y+1}{3y-2}.$$

In this equation, we may replace y with x if we wish.

The domain of f is all x for which the expression $\frac{2x+1}{3x-4}$ is defined, and, in particular, the only issue is when $3x-4$ equals 0. Therefore, the domain is all real numbers except for $x = \frac{4}{3}$.

The domain of f^{-1}, similarly, is all real numbers except for $x = \frac{2}{3}$.

3. (5 points). A particular capacitor stores an electric charge of $Q(t) = Q_0(1 - e^{-t/3})$ while recharging, where Q_0 is the capacitor's maximum capacity in Coulombs and t is in seconds. How long does it take for the capacitor to recharge to 90% of its capacity, starting from zero charge?

   We are wanting to find how long it takes for the capacitor to go from 0 to $0.9Q_0$. Just to check, we see that $Q(0) = Q_0(1 - e^{-0/3}) = 0$, so we may write the equation as

$$0.9Q_0 = Q(t)$$

   or

$$0.9Q_0 = Q_0(1 - e^{-t/3}).$$

   Dividing both sides by Q_0 and then solving for t, we have:

$$\begin{aligned} 0.9 &= 1 - e^{-t/3} \\ e^{-t/3} &= 0.1 \\ \frac{-t}{3} &= \ln 0.1 \\ -t &= 3\ln 0.1 \\ t &= -3\ln 0.1 \end{aligned}$$

   We may simplify this by using the fact that $-3\ln 0.1 = \ln 0.1^{-3} = \ln 1000$.