# Precalc Review

1. What is $1 + 2 \cdot 3$? What is $(1 + 2) \cdot 3$?

   Just remember order of operations: $1 + 2 \cdot 3 = 1 + (2 \cdot 3) = 7$. The other one is 9.

2. Why are $(-3)^2$ and $-3^2$ different? And what are their values?

   The negative sign has lower precedence than exponentiation. $(-3)^2 = (-3) \cdot (-3) = 9$, whereas $-3^2 = -3 \cdot 3 = -9$.

3. Simplify $\left(\frac{2}{\sqrt{2}}\right)^{-4}$ and $\sqrt{200} - \sqrt{32}$.

   Remember that a negative exponent flips fractions, so $(\frac{2}{\sqrt{2}})^{-4} = (\frac{\sqrt{2}}{2})^4 = \frac{\sqrt{2}^4}{2^4} = \frac{4}{16} = \frac{1}{4}$.

   Remember that $\sqrt{ab} = \sqrt{a}\sqrt{b}$, so $\sqrt{200} - \sqrt{32} = \sqrt{100 \cdot 2} - \sqrt{16 \cdot 2} = \sqrt{100}\sqrt{2} - \sqrt{16}\sqrt{2} = 10\sqrt{2} - 4\sqrt{2} = (10 - 4)\sqrt{2} = 6\sqrt{2}$.

4. Simplify $(3a^3b^3)(4ab^2)^2$.

   $(3a^3b^3)(4ab^2)^2 = (3a^3b^3)(4ab^2)(4ab^2)$. Since you can do multiplication in any order, this is $(3 \cdot 4 \cdot 4)(a^3aa)(b^3b^2b^2) = 48a^5b^7$.

5. Expand and simplify $(\sqrt{a} - \sqrt{b})(\sqrt{a} + \sqrt{b})$ and $(x + 2)^3$.

   The first is easy if you remember $(x - y)(x + y) = x^2 - y^2$, substituting $x = \sqrt{a}$ and $y = \sqrt{b}$. Or, you can just multiply it out. In either case, you get $a - b$.

   For the second one, you can remember the binomial theorem, or you can do long multiplication:

$$\begin{aligned}
(x+2)^3 &= (x+2)(x+2)^2 \\
&= (x+2)(x(x+2) + 2(x+2)) \\
&= (x+2)((x^2+2x) + (2x+4)) \\
&= (x+2)(x^2+4x+4) \\
&= x(x^2+4x+4) + 2(x^2+4x+4) \\
&= (x^3+4x^2+4x) + (2x^2+8x+8) \\
&= x^3+6x^2+12x+8
\end{aligned}$$

6. Factor $x^3y - 4xy$.

   Notice $x^3y = x^2(xy)$, so this is $xy(x^2 - 4)$. But, $x^2 - 4$ also can be factored, which you can either see because of the formula $(a - b)(a + b) = a^2 - b^2$ with $a = x$ and $b = 2$, or because you notice that $x^2 - 4$ has $\pm 2$ as roots. So, it's $xy(x - 2)(x + 2)$.

7. Simplify $\frac{x^2}{x^2-4} - \frac{x+1}{x+2}$.

   This uses the multiplication by 1 trick, with a somewhat complicated version of 1. First, you have to notice that $x^2 - 4 = (x - 2)(x + 2)$ (like in the previous problem), and then multiply the right fraction by $\frac{x-2}{x-2}$ (which is 1 whenever $x \neq 2$, which is ok because the original expression isn't defined when $x = 2$ anyway). So, it's

$$\begin{aligned}
&= \frac{x^2}{(x-2)(x+2)} - \frac{x+1}{x+2} \cdot \frac{x-2}{x-2} \\
&= \frac{x^2}{(x-2)(x+2)} - \frac{(x+1)(x-2)}{(x+2)(x-2)} \\
&= \frac{x^2 - (x+1)(x-2)}{(x-2)(x+2)} \\
&= \frac{x^2 - (x^2 - x - 2)}{(x-2)(x+2)} \\
&= \frac{x+2}{(x-2)(x+2)} \\
&= \frac{1}{x-2}
\end{aligned}$$

(whenever $x \neq \pm 2$).

8. Find all $x$ such that $x^4 - 3x^2 + 2 = 0$.

   Notice you can write this as $(x^2)^2 - 3(x^2) + 2 = 0$, so use the quadratic formula, solving for $x^2$:

$$\begin{aligned}
x^2 &= \frac{3 \pm \sqrt{(-3)^2 - 4 \cdot 1 \cdot 2}}{2 \cdot 1} \\
&= \frac{3 \pm \sqrt{1}}{2} \\
&= 1, 2
\end{aligned}$$

   And then solving for $x$, you have $x = \pm 1, \pm\sqrt{2}$ (always remember both the positive and negative value of a square root!).

   Another way of dealing with this is to notice you can factor it as $(x^2 - 1)(x^2 - 2) = 0$ and then solving each quadratic separately.

9. Give examples which show that the following are just wishful thinking:

   (a) $(a + b)^2 = a^2 + b^2$
   (b) $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$
   (c) $\sqrt{a^2 + b^2} = a + b$
   (d) $\frac{1}{a+b} = \frac{1}{a} + \frac{1}{b}$

   (and find some values where the "rules" work).

   (a) $(2 + 2)^2 \neq 2^2 + 2^2$, but $(0 + 0)^2 = 0^2 + 0^2$.

(b) $\sqrt{2+2} \neq \sqrt{2} + \sqrt{2}$, but $\sqrt{0+0} = \sqrt{0} + \sqrt{0}$.

(c) $\sqrt{2^2+2^2} \neq 2+2$, but $\sqrt{0^2+0^2} = 0+0$.

(d) $\frac{1}{2+2} \neq \frac{1}{2} + \frac{1}{2}$. It actually can never work: if it could, we would need $\frac{1}{a+b} = \frac{1}{a} + \frac{1}{b} = \frac{a+b}{ab}$, and so $ab = (a+b)^2$, which gives $a^2 + ab + b^2 = 0$. But, using the quadratic formula to solve for $a$, we have $a = \frac{-b \pm \sqrt{b^2 - 4b^2}}{2}$, however $b^2 - 4b^2 = -3b^2$ is always negative (when $b$ is real).

10. Find an equation for the line with slope 2 which passes through $(2, 2)$.

An equation for such a line is $y - y_0 = m(x - x_0)$, where $m = 2$ and $(x_0, y_0) = (2, 2)$. To remember this, look at this as shifting a line $y = mx$ to pass through $(x_0, y_0)$ rather than $(0, 0)$.

11. Where does the line $x + y = 1$ intersect the circle $x^2 + y^2 = 2$?

Use the first equation to get $y = 1 - x$, and substitute into the circle's equation:

$$\begin{aligned} 2 &= x^2 + (1-x)^2 \\ &= x^2 + 1 - 2x + x^2 \\ &= 2x^2 - 2x - 1 \end{aligned}$$

which is to say $0 = 2x^2 - 2x - 1$. Use the quadratic formula to get $x = \frac{2 \pm \sqrt{2^2 - 4 \cdot 2 \cdot (-1)}}{2 \cdot 2} = \frac{2 \pm 2\sqrt{3}}{4} = \frac{1 \pm \sqrt{3}}{2}$. Then, to get $y$, we have $y = 1 - x = 1 - \frac{1 \pm \sqrt{3}}{2} = \frac{1 \mp \sqrt{3}}{2}$. So, the two points are $(\frac{1+\sqrt{3}}{2}, \frac{1-\sqrt{3}}{2})$ and $(\frac{1-\sqrt{3}}{2}, \frac{1+\sqrt{3}}{2})$. To check this answer, notice that both the line and the circle are symmetric about the line $y = x$ (draw a quick sketch!), and these two points are also symmetric in this way.

12. Let $f(x) = x^3$. Evaluate $\frac{f(2+h) - f(2)}{h}$.

The expansion of $(h+2)^3$ is useful here, and luckily we did it back in problem 5.

$$\begin{aligned} \frac{f(2+h) - f(2)}{h} &= \frac{(2+h)^3 - 2^3}{h} \\ &= \frac{h^3 + 6h^2 + 12h + 8 - 8}{h} \\ &= \frac{h(h^2 + 6h + 12)}{h} \\ &= h^2 + 6h + 12. \end{aligned}$$

13. For $f(x) = 1 + (x-1)^{-1}$, make a rough sketch of the graph of $y = f(x)$.

Shift the graph of $1/x$ one unit up (in the positive $y$ direction) and one unit to the right (in the positive $x$ direction). All you really need to capture are the asymptotes $x = 1$ and $y = 1$ and when the graph is above or below $y = 1$.

14. What is the domain of the function $f(x) = \frac{2x+1}{x^2+x-2}$? Of $g(x) = 2\sqrt{1+x}$?

    The function $f$ is defined whenever $x^2 + x - 2 \neq 0$. When is $x^2 + x - 2 = 0$ then? By the quadratic formula or factoring, we see $x = -2, 1$, so the domain is $(-\infty, -2) \cup (-2, 1) \cup (1, \infty)$ (that is, a union of open intervals so that we have everything but $-2$ and 1).

    For $g$, we're assuming we're dealing only with functions with real number inputs and outputs. Since the domain of $\sqrt{\cdot}$ is all non-negative real numbers, the domain of $g$ is $[-1, \infty)$ (or, in other words, all $x \geq -1$).

15. What is the maximum value of the function $f(x) = 1 - 2x - x^2$? What about for $g(x) = \frac{2}{x^2-2x+3}$?

    First, a sanity check: The $x^2$ term is negative, so it ultimately will make the function more and more negative faster than $1 - 2x$ can make it positive, so there ought to be a maximum value it takes on. With that out of the way, we complete the square: $1-2x-x^2 = -(x+1)^2+2$, so $f$ has a maximum of 2 at $x = -1$. (Notice that $-(x+1)^2$ has a maximum of 0 at $x = -1$, and it is negative everywhere else.)

    For $g$, first let's complete the square for $h(x) = x^2 - 2x + 3$. This is $(x-1)^2 + 2$, which has a minimum value at $x = 1$ which is 2 (so $h$ is always positive). Since $g(x) = 2/h(x)$ (which also must always be positive), there is a maximum at $x = 1$ which is 1. Think about why taking reciprocals basically "flips" an always-positive function.

16. Which is bigger? $1000^{10}$ or $10^{1000}$? (Try using logarithms to prove it.)

    First, the logarithm function is increasing, so $a \leq b$ if and only if $\log a \leq \log b$. This means we can compare them by comparing their logarithms. For the first, $\log 1000^{10} = 10 \log 1000 = 10 \log 10^3 = 30 \log 10$. For the second, $\log 10^{1000} = 1000 \log 10$. The second one is clearly larger by quite a bit.

    Another way of dealing with it, without logarithms, is that $1000^{10} = (10^3)^{10} = 10^{30}$, and this is clearly less than $10^{1000}$.

17. Compute $\sin^2(72°) + \cos^2(72°)$.

    This is computable with some amount of work (good practice), or you can remember the identity $\cos^2 x + \sin^2 x = 1$.

18. Remember $\sin\theta$ and $\cos\theta$ when $\theta = 0, \pi/6, \pi/4, \pi/3, \pi/2$, and plot each $(\cos\theta, \sin\theta)$.

    The points should trace out a quarter of a circle (because of the Pythagorean theorem, which really is what the above identity is about). You should memorize these values if you haven't already.

19. Remember the first few digits of $\sqrt{2}$, $1/\sqrt{2}$, and $\sqrt{3}/2$.

    These are also good to memorize.

    $\sqrt{2} = 1.414\ldots$ $\quad$ $1/\sqrt{2} = \sqrt{2}/2 = 0.707\ldots$ $\quad$ $\sqrt{3}/2 = 0.866\ldots$

For more review, take a look at xxiv-xxviii in Stewart.