Research statement. Kyle Miller

My research revolves around low-dimensional topology, representation theory, and some combinatorics. I am interested in finding algebraic interpretations of diagrams (for example, diagrams of graphs and knots) and diagrammatic interpretations of algebra (for example, tensor networks and diagram categories). My main research programs are:

1. Taking invariants of graphs and ribbon graphs from combinatorics and reinterpreting them from the perspective of TQFTs. This entails finding ways to cut up the invariant so it may be computed in a piece-by-piece manner using suitable algebraic structures.

2. Finding new invariants of knots, links, and spatial graphs in 3-manifolds, for example thickened surfaces up to stabilization (virtual knots and virtual spatial graphs) using techniques from the first program.

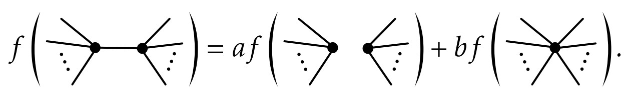
I am also interested in using computers to help with mathematical research, for example writing Mathematica packages [MSM], creating tools to identify knots and to compute tables of invariants [Mil20], and formalizing mathematics in the Lean proof assistant.

1 Topological graph polynomials and 2D TQFT

Background. Many interesting ring-valued graph invariants f satisfy a *deletion-contraction relation*, which for fixed constants a and b is a linear relation of the form

$$f(G) = af(G-e) + bf(G/e)$$

that holds for all graphs G and edges e of a particular type, such as non-loop or non-bridge edges, and we may diagrammatically regard this as a sort of skein relation:

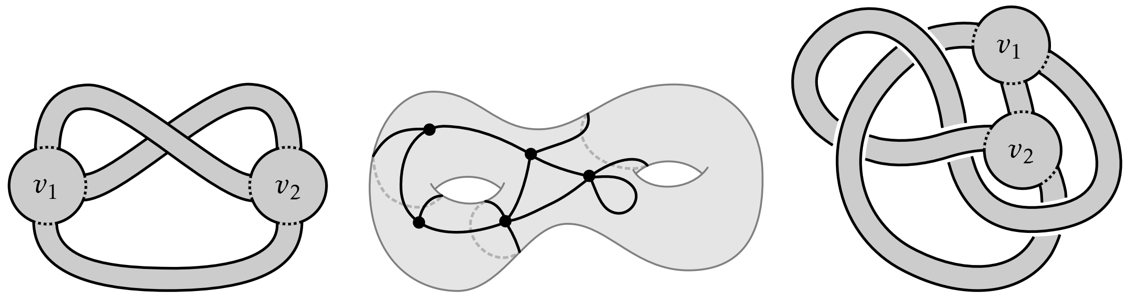


Examples include the chromatic polynomial [Bir13, Whi32], the flow polynomial [Tut47], the partition function of the Q-states Potts model in statistical physics [Wu82], and the Jones polynomial of an alternating knot from its Tait graph [Thi87]. Tutte initiated the study of graph invariants satisfying deletion-contraction relations [Tut47, Tut54], and he defined a two-variable polynomial that is a "universal" deletion-contraction invariant known as the *Tutte polynomial*.

There are a number of other graph-like objects with deletion-contraction invariants:

- A *ribbon graph* (Figure 1a) is a topological realization of a graph as an (oriented) surface, where vertices correspond to disks and edges to rectangular strips. The *Bollobás–Riordan polynomial* [BR01, BR02] is a three-variable polynomial that is a deletion-contraction invariant that generalizes the universality property of the Tutte polynomial to ribbon graphs. An application in knot theory is that a specialization of the Bollobás–Riordan polynomial of the A-state Turaev ribbon graph of a link is its Jones polynomial [DFK⁺08], and the polynomial was used to relate Khovanov homology to Turaev surfaces [CK14].

- A *surface graph* (Figure 1b) is a graph embedded in a closed (oriented) surface. Similar "state sum" approaches have been used to define deletion-contraction surface graph invariants such as the *Krushkal polynomial* [Kru11] and the *surface Tutte polynomial* [GKRV16].



(a) A ribbon graph. (b) A surface graph. (c) A spatial graph.

Figure 1: Other graph-like objects.

- A *spatial graph* (Figure 1c) is an embedding of a ribbon graph in a 3-manifold. The *Yamada polynomial* [Yam89] is a spatial graph invariant that is the Reshetikhin–Turaev invariant where each edge is colored with the 3-dimensional irreducible representation of $\mathcal{U}_q(\mathfrak{sl}(2))$.

For polynomial-valued deletion-contraction invariants, there are two coincidental phenomena: (1) where real roots tend to cluster and (2) values at which the invariant has exceptional linear relations (that is, additional skein relations). The first is motivated by statistical physics, where the roots of the partition function of the Q-states Potts model play a role in the theory of phase transitions, and Beraha observed roots for planar lattices tend to cluster at *Beraha numbers*.

While (1) has not yet been adequately explained, a complete answer for (2) is known for the chromatic and flow polynomial of planar graphs [KS93,FK09,FK10]. The polynomials can be calculated in the Temperley–Lieb category, and exceptional linear relations arise precisely at values for which the trace radical of the category is nontrivial.

I give a complete answer to (2) for the Tutte polynomial and for the Bollobás–Riordan polynomial along the locus for which it has its duality relation.

1.1 Functorial deletion-contraction invariants of ribbon graphs

The success of this representation-theoretic approach in explaining exceptional linear relations suggested a program to construct appropriate categories such that graph invariants extend to symmetric monoidal functors ("TQFTs"). There is some overlap between this and the Morrison–Peters–Snyder program to classify categories based on their linear relations [MPS17].

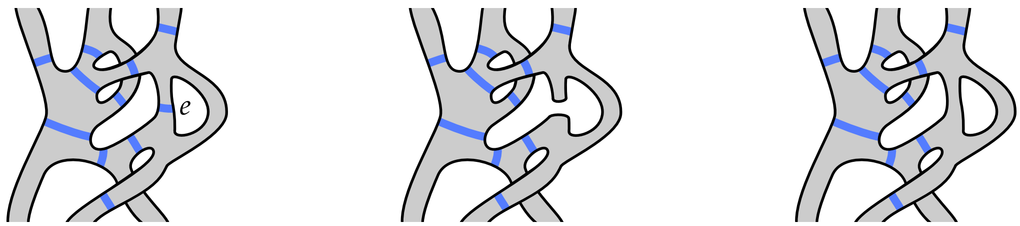
The chromatic category in [FK09] was defined in terms of the flow polynomial because it has a convenient property: it is invariant under edge subdivisions, hence there are natural choices for identity morphisms and traces. For general ribbon graphs, these are cause for complication.

In a forthcoming paper, *A category in graph theory*, I define a compact closed category $\mathscr{R}$ for ribbon graphs by generalizing them to allow vertices to be arbitrary compact oriented surfaces with nonempty boundary, which is justified because every deletion-contraction invariant extends to such generalized ribbon graphs in a unique way. For k a field, symmetric monoidal functors $\mathscr{R} \to k$-**Mod** that satisfy a deletion-contraction relation correspond to symmetric Frobenius algebras, and the collection of functors associated to the semisimple Frobenius algebras determines the most-universal deletion-contraction ribbon graph invariant.

The Bollobás–Riordan polynomial arises from the special Frobenius algebras (parameterized by $\mathrm{Mat}_n(k) \otimes k^m$) with rescaled counits, giving a three-variable (Laurent) polynomial, and the specialization $n = 1$ gives the Tutte polynomial. The locus along which the Bollobás–Riordan polynomial satisfies its celebrated duality relation coincides precisely with the case $m = 1$.

The ribbon graph category

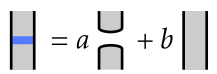
In more detail, the category $\mathscr{R}$ is derived from the category $\mathbf{2Cob}^{\text{open}}$ of *open cobordisms* [LP08]. Objects of $\mathbf{2Cob}^{\text{open}}$ are disjoint unions of oriented closed intervals, and morphisms are compact oriented 2D cobordisms with corners whose components have nonempty boundary. Then, $\mathscr{R}$ is $\mathbf{2Cob}^{\text{open}}$ but morphisms are decorated with disjoint properly embedded arcs called *edges*. Edge deletion corresponds to removing a neighborhood of the arc from the surface, and edge contraction to forgetting the arc. Figure 2c is an example of edge contraction giving a non-disk vertex.



(a) A morphism G in $\mathscr{R}$. (b) The deletion $G-e$. (c) The contraction G/e.

Figure 2: Deletion and contraction of an edge in a morphism in $\mathscr{R}$.

For R a ring and $a, b \in R$ constants, we can form a category $\mathscr{R}_{a,b}$ by enriching $\mathscr{R}$ over R and quotienting by the following linear relation in $\mathscr{R}(I, I)$, with I the interval object:



We then study symmetric monoidal functors $\mathscr{R} \to R\text{-}\mathbf{Mod}$ that factor through some $\mathscr{R}_{a,b}$. The category is equivalent to the R-enrichment of $\mathbf{2Cob}^{\text{open}}$, and functors from $\mathbf{2Cob}^{\text{open}}$ correspond to symmetric Frobenius R-algebras.

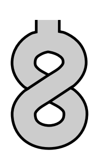
Classification of deletion-contraction invariants

At least for now, a complete classification of symmetric Frobenius algebras appears to be intractable, however we say two symmetric Frobenius algebras are *graphically equivalent* if they define the same ribbon graph invariant.

Theorem 1.1. *If k is a field of characteristic 0, then a symmetric Frobenius algebra is graphically equivalent to a direct sum of matrix rings and, possibly, $k[x]/(x^n)$, with appropriate counits.*

On the way to this, I studied a decomposition of general Frobenius algebras. Related to $\mathbf{2Cob}^{\text{open}}$ is $\mathbf{2ImmCob}^{\text{open}}$, of cobordisms along with an orientation-preserving immersion in the plane, and symmetric monoidal functors $\mathbf{2ImmCob}^{\text{open}} \to k\text{-}\mathbf{Mod}$ correspond to Frobenius k-algebras. Similar to the commutative case [Abr97], I show:

Lemma 1.2. *If A is a Frobenius algebra over a field k, then there is a morphism of $\mathbf{2ImmCob}^{\text{open}}(\emptyset, I)$ whose image $\omega \in A$, called the* distinguished element, *generates the socle of A. It is given by*

$\omega =$  .

The distinguished element can be used to extract the direct summands of the [Hal40, Jan59] decomposition $A = B \oplus B^\perp$, where B is the largest semisimple subalgebra and the orthogonal complement $B^\perp$ is a *radicular* algebra, meaning the socle of $B^\perp$ is a subset of its Jacobson radical.

Theorem 1.3. *If A is a Frobenius algebra over a field k of characteristic 0, then $B = A\omega^2$ in the above decomposition.*

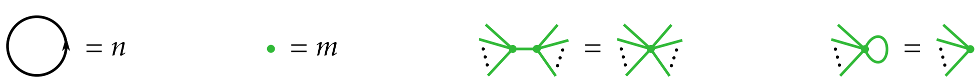
While the Artin–Wedderburn theorem classifies semisimple Frobenius algebras, this result implies that radicular symmetric Frobenius algebras contribute a $k[x]/(x^n)$ in Theorem 1.1.

The semisimple Frobenius k-algebras determine the value of a graph in $\mathscr{R}_{a,b}(\emptyset,\emptyset)$ in the sense that they give an embedding $\mathscr{R}_{a,b} \to \prod_A k$ where A ranges over these algebras. The semisimple Frobenius k-algebras are parameterized via the Artin–Wedderburn theorem by the number of matrix direct summands m, the sizes of the matrices $\mathbf{n} = (n_1, \ldots, n_m)$, and the nonzero constants $\mathbf{c} = (c_1, \ldots, c_m)$ that scale the respective direct summand's trace form. The value of a connected surface $\Sigma \in \mathbf{2Cob}^{\mathrm{open}}(\emptyset,\emptyset)$ through the corresponding functor is a Laurent polynomial $\sigma_{b_1(\Sigma),g(\Sigma)}$ in the components of $\mathbf{n}$ and $\mathbf{c}$, and the image of $\mathscr{R}_{a,b}(\emptyset,\emptyset)$ in $\prod_A k$ as A ranges over semisimple Frobenius k-algebras can be regarded as an element of $k[\sigma_{i,j} \mid i,j \in \mathbb{N} \text{ and } i \geq 2j]$. These symmetric functions are generically algebraically independent, and $\sigma_{i,j}$ can be identified with a surface of genus j with $i - 2j + 1$ punctures, hence the image is isomorphic to $\mathscr{R}_{a,b}(\emptyset,\emptyset)$.

Identifying the Bollobás–Riordan polynomial

The symmetric Frobenius algebra $A = \mathrm{Mat}_n(k) \otimes k^m$ with counit $x \otimes e_i \mapsto c \operatorname{tr} x$ gives an $\mathscr{R}_{1,1}$ ribbon graph invariant that is equivalent to the Bollobás–Riordan polynomial, and it has a convenient two-layer graphical calculus to calculate it. Edges and vertices of ribbon graphs are expanded as

where the black curves correspond to $\mathrm{Mat}_n(k) \approx (k^n)^* \otimes k^n$ and the green graph to k^m, and these diagrams are subject to the following relations:



The Tutte polynomial is at $n = 1$, and the two-variable specialization at which the Bollobás–Riordan polynomial has its duality relation is at $m = 1$.

Each Kauffman state of a link diagram corresponds to the boundary of a corresponding state in the above expansion for the A-state Turaev ribbon graph, and hence $(c,n,m) = (A^{-2}, -A^2 - A^{-2}, 1)$ is the Kauffman bracket times a power of A depending on the diagram. Thus [DFK+08] follows.

Enumerating exceptional linear relations for the Tutte polynomial

The commutative Frobenius algebra k^m as the standard represetation for S_m gives a symmetric monoidal functor that plays the role in providing centralizer algebras for the Schur–Weyl duality of $\mathrm{End}_{S_m}(k^m \otimes \cdots \otimes k^m)$. Hence, it is intimately related with the *partition categories*, whose representation theory is very well understood [HR05]. Partition algebras were originally devised by Martin and Jones for computing partition functions of the Q-states Potts model for nonplanar graphs [Mar90, Mar94, Jon94, Mar96], and Deligne defined a category $\mathrm{Rep}(S_t)$ for generic t that interpolates between the pseudo-abelian envelopes of the partition categories [Del07]. The category has a nontrivial trace radical if $t \in \mathbb{N}$, and the central idempotents that generate it yield a complete list of exceptional linear relations for the Tutte polynomial of nonplanar graphs.

The flow polynomial's deletion-contraction relation is a scalar multiple of a projection onto $V_{(m-1,1)} \subset k^n$, and the corresponding centralizer algebra is known as the *quasi-partition algebra* [DO14]. This gives a more precise list of exceptional linear relations, solving the issue that the projector annihilates many of the linear relations derived from the partition algebra.

Calvin McPhail-Snyder and I have created a Mathematica package defining categories such as the partition category, and it contains code for computing the central idempotents [MSM].

Proposed research

- The radicular Frobenius algebra Λk^2 is commutative in the category of graded vector spaces, and its graph invariant there counts spanning forests. While this invariant already appears as an evaluation of the Tutte polynomial and as an evaluation of a two-variable polynomial associated to $k[x]/(x^n)$, it is not directly representable with a symmetric Frobenius algebra in k-**Mod**. Analyzing symmetric radicular Frobenius algebras in suitable categories would allow invariants to be represented exactly, which has applications in algebraic combinatorics.

- While the symmetric Frobenius algebra $A = \mathrm{Mat}_n(k) \otimes k^m$ is a representation for $G = \mathrm{GL}(k^n) \times S_m$, ribbon graphs do not surject onto the centralizer algebra for $\mathrm{End}_G(A \otimes \cdots \otimes A)$, hence the representation theory of $\mathrm{GL}(k^n)$ and S_m does not immediately give the exceptional linear relations for the Bollobás–Riordan polynomial. Still, experimental evidence suggests exceptional relations occur exactly when $n \in \mathbb{Z}$ or $m \in \mathbb{N}$, and so enumeration of exceptional linear relations would follow from identification of the correct commutant G.

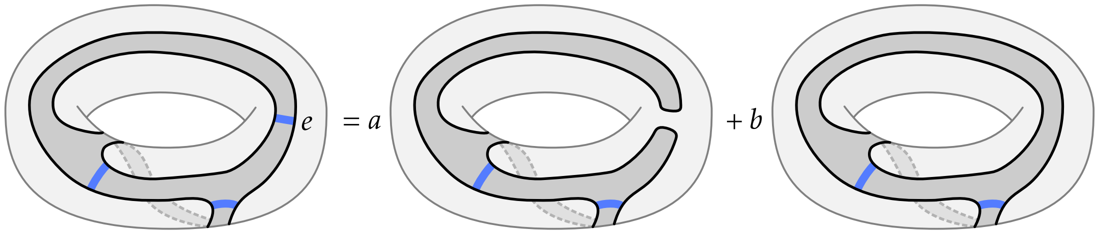
- There is an obvious analogue to $\mathscr{R}$ for unoriented ribbon graphs, and symmetric monoidal functors from this category correspond to symmetric Frobenius algebras with an antiautomorphism. The study of such objects should yield a functorial interpretation of the four-variable Bollobás–Riordan polynomial for unoriented ribbon graphs.

- The representation k^m in $\mathrm{Rep}(S_m)$ splits as $V_m \oplus V_{(m-1,1)}$, and the Tutte polynomial may be regarded as a state sum from coloring edges with xV_m or $V_{(m-1,1)}$, where x denotes a weighting factor, and at $x = 0$ this is equivalent to the flow polynomial. The vanishing locus of this as a polynomial in x and m tends to cross the m axis orthogonally (which in the usual parameterization appears as hyperbolae). This phenomenon should have some bearing on the planar and non-planar Beraha conjectures, perhaps by a [Sok05] multivariate approach.

1.2 Fully extended black-white 2D TQFTs

We can think of a surface graph as being a ribbon graph embedded in a closed surface, and, just like for ribbon graphs, we can safely generalize surface graphs to allow for non-disk vertices. We may regard generalized surface graphs as being closed (oriented) surfaces partitioned into a black region and a white region along with a collection of arcs properly embedded in the black region.

Graphically, we can depict $[G] = a[G - e] + b[G/e]$ like so:



Edgeless surfaces like these may be regarded as 2D cobordisms with 1D defects, where there are black and white "world sheets" and a single kind of "domain wall" between them [DKR].

Calling this cobordism category **BW2Cob**, the state sum models of [FHK, LP] extend to constructions for TQFTs **BW2Cob** $\to k$-**Mod** by using a pair of strongly separable Frobenius algebras B and W over k, and this is by treating the black and white regions as separate copies of $\mathbf{2Cob}^{\mathrm{ext}}$, the category of open-closed cobordisms. The Krushkal polynomial arises from using a copy of the Bollobás–Riordan Frobenius algebra in each region, illuminating its duality relation.

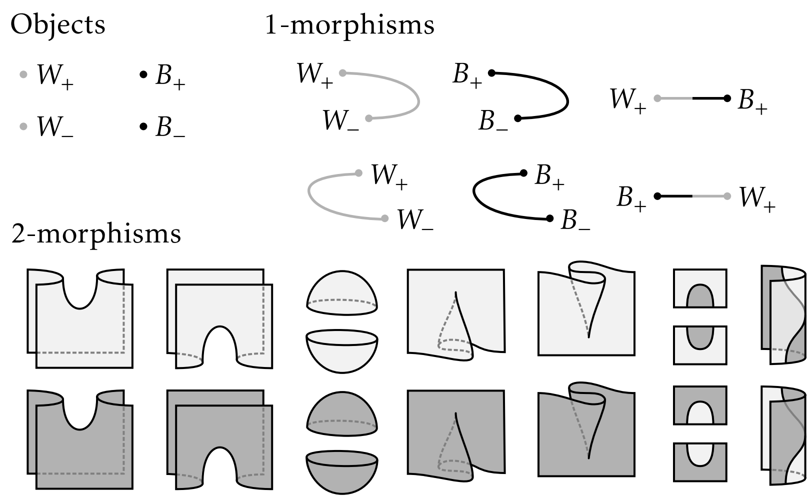


Figure 3: The expected generators for the symmetric monoidal bicategory **BW2Cob**.

Proposed research

- The category **BW2Cob** is infinitely-generated, and a way to deal with this is to consider a 2-category of fully extended black-white 2D TQFTs. Fully extended 2D TQFTs were classified in [SP] using a refinement to Morse and Cerf theory to handle generic composites $\Sigma \to \mathbb{R}^2 \to \mathbb{R}$ for Σ a surface. For black-white surface with black region $B \subseteq \Sigma$, this would involve incorporating genericity for ∂B so that the image of ∂B in $\mathbb{R}^2$ is an immersed curve in generic position. With this in mind, Figure 3 is an expected set of generators for **BW2Cob**. Relations would be from more careful study of generic maps $\Sigma \times I \to \mathbb{R}^2 \times I$.

  One would expect that a fully extended **BW2Cob** TQFT corresponds to a pair of Morita contexts of strongly separable Frobenius algebras along with Morita contexts between corresponding algebras in each pair.

  It should be the case that the *surface Tutte polynomial* in [GKRV16] corresponds to symmetric functions on the parameters from the Artin–Wedderburn decomposition of the involved semisimple Frobenius algebras when over $\mathbb{C}$.

1.3 Penrose polynomials

Penrose defined invariants of trivalent ribbon graphs associated to metric Lie algebras $\mathfrak{g}$ [Pen71, BN97]. As a tensor network, the vertices are assigned the rotationally-invariant 3-form $([-,-],-)$, contracted along edges via the metric. When $\mathfrak{g}$ is $\mathfrak{so}(N)$, $\mathfrak{sl}(N)$, or $\mathfrak{sp}(2N)$, the invariant is a polynomial in N, and the polynomials contain information such as planarity of the underlying graph.

For $\mathfrak{so}(N)$, [EMM13] gives an extension to general unoriented ribbon graphs with a deletion-contraction-like relation. Using the Brauer category, McPhail-Snyder and I showed in

> [MSM20] *Planar diagrams for local invariants of graphs in surfaces*, J. Knot Theory Ramifications **29** (2020), no. 1, 1950093, 49, arXiv:1805.00575v2 [math.GT].

that the $\mathfrak{sl}(N)$ polynomial extends to general ribbon graphs with signed vertices and a deletion-contraction-like relation. In a forthcoming paper, I show how there is a two-variable polynomial of ribbon graphs with signed vertices that satisfies a deletion-contraction-like relation and specializes to the $\mathfrak{so}(N)$, $\mathfrak{sl}(N)$, and $\mathfrak{sp}(2N)$ Penrose polynomials. By thinking of a negative-dimensional space as a super vector space in grading 1, then the $\mathfrak{sp}(2N)$ polynomial is the "$\mathfrak{so}(-2N)$" polynomial. The polynomial can be understood functorially using diagrams for Deligne's $\mathrm{Rep}(\mathrm{O}(t))$, and underlying the construction is that $\mathfrak{gl}(N) = \mathrm{Mat}_N$ is a Frobenius algebra.

Proposed research

- The Penrose invariants of other metric Lie algebras have also been studied. If they arise as the Lie algebra associated to a Frobenius algebra, then one might be able to form a universal Penrose polynomial that generalizes to non-trivalent ribbon graphs while satisfying a deletion-contraction-like relation.

- I have calculated $\mathfrak{sl}(N)$ polynomials for all cubic connected graphs with up to 22 vertices and girth at least 3. Experimentally, the roots of the "$\mathfrak{sl}(x^{-1/2})$" polynomial near 1/4 are all real, and complex roots appear to be clustered on circles centered at 1/4. By investigating the roots of the two-variable polynomial, or by taking a Sokal-like multivariate approach [Sok05], this phenomenon might be better understood.

1.4 Graph (co)homology

Since the success of Khovanov homology in categorifying the Jones polynomial [Kho], there have been many similar categorifications of other state-sum invariants such as the chromatic polynomial and Tutte polynomial [HGR05, JHR06, Sto08].

The functorial point of view gives a framework for categorifications. For R a ring, consider the bicategory Alg^2_R of R-algebras, bimodules, and intertwiners. Symmetric monoidal functors $[\![-]\!] : \mathbf{2Cob}^{\mathrm{open}} \to \mathrm{Alg}^2_R$ correspond to symmetric Frobenius objects, which consist of an R-algebra A along with a collection of bimodules ${}_AM_{A\otimes A}$, ${}_{A\otimes A}\Delta_A$, ${}_A\eta$, and ε_A satisfying Frobenius algebra axioms such as $M \otimes_{A\otimes A} (\eta \otimes_R A) \cong A$, and so on.

Generalized, [HGR05] is the following. For $R = \mathbb{Z}$ and A a commutative ring, taking all of the bimodules to be A gives a commutative Frobenius object. Pass to the category of cochain complexes of bimodules. Multiplication in A defines a bimodule map $\mu : \eta \otimes_R \varepsilon \to A$ which in turn is used to define a bracket $[\![G]\!] = \mathrm{Cone}([\![G-e]\!] \to [\![G/e]\!])$. The cochain complex $[\![G]\!]$ up to isomorphism is a graph invariant, its Euler characteristic is the chromatic polynomial evaluated at $\dim(A\otimes\mathbb{Q})$, and deletion-contraction gives a long exact sequence of cohomology groups.

If a symmetric Frobenius object A in Alg^2_R is a bialgebra, then the counit for the bialgebra gives an A-bimodule map $A \to \eta \otimes_R \varepsilon$, and using this to define $[\![G]\!] = \mathrm{Cone}([\![G/e]\!] \to [\![G-e]\!])$ gives another chain-complex-valued invariant. If $A = \mathbb{Z}[\Gamma]$ for Γ a finite abelian group the Euler characteristic is the flow polynomial evaluated at $|\Gamma|$.

Cones do not so easily extend to a functor for all of $\mathscr{R}$, but to fix this we can pass to the derived category, though now calculations are at least as difficult as Hochschild homology. Short exact sequences involving A and $\eta \otimes_R \varepsilon$ give exact triangles for a deletion-contraction relation.

Proposed research

- In the case of A a bialgebra, these constructions are functorial on graph minors, hence it would be worth considering excluded minors theorems. For example, it might be the case that non-planarity is detected by a cohomology class from $K_{3,3}$ or K_5.

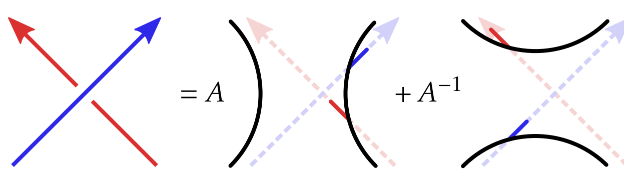
2 Invariants of virtual knots and spatial graphs

Background. Virtual links are links in thickened surfaces up to stabilization by vertical annuli [Kau99, Kup03]. In a sense, virtual links capture some notion of the local topology of surface links. Virtual spatial graphs are ribbon graphs in thickened surfaces, again up to stabilization.

2.1 The arrow polynomial

The Kauffman bracket has a generalization to framed virtual links called the *arrow polynomial*, which is a polynomial in $\mathbb{Z}[A^{\pm 1}, X_1, X_2, \dots]$ [DK09]. The polynomial is defined by keeping track of "uncanceled" cusps that arise from the two smoothings of the Kauffman bracket.

In *The homological arrow polynomial for virtual links* (draft on my website), I generalize the arrow polynomial for framed virtual links L to be an invariant in $R = \mathbb{Z}[A^{\pm 1}][X_{\pm J} \mid J \in H_0(L)]$, where we set $X_0 = 1$. Considering L in a thickened surface $\Sigma \times I$, there is a map $j : H_1(\Sigma) \to H_0(L)$ by taking intersection numbers with L projected to the boundary. This map extends to the Kauffman bracket skein module $\mathrm{Sk}(\Sigma \times I) \to R$ by multiplicatively sending simple loops S to $(-A^2 - A^{-2}) X_{\pm j(S)}$, where the $\pm$ takes into account the fact that S is not oriented. The image of $[L] \in \mathrm{Sk}(\Sigma \times I)$ through this map is the *homological arrow polynomial*. There is a graphical calculus for this by recording local contributions to intersection numbers with colored whiskers:



Whiskers commute, and whiskers of the same color pointing in opposite directions cancel. The net numbers of whiskers on a state loop gives an element in $\pm H_0(L)$.

The homological point of view for the arrow polynomial has immediate applications for null-homologous virtual links, and in particular checkerboard-colorable links. By using the 2-cabled arrow polynomial, I was able to complete Imabeppu's characterization of checkerboard colorability of all virtual knots up to four crossings.

For a natural notion of *reduced*, the Kauffman–Murasugi–Thistlethwaite theorem was generalized to alternating virtual knots in [BK] by defining a Kauffman bracket that incorporates

Krushkal polynomial terms, strengthening a previous generalization by [Kam] that used the Kauffman bracket alone, but the polynomial depends on a minimal-genus representative. Using the homological arrow polynomial, which does not need a minimal-genus representative, I proved a generalization that is of roughly intermediate strength. A diagram is *h-reduced* if every simple closed curve in the surface that transversely intersects only a single crossing does so with non-zero intersection number. I showed that the A-breadth of the polynomial for an h-reduced diagram D is $4(c(D) - g(D))$, where $g(D)$ is the genus of the diagram as a ribbon graph.

I have implemented the cabled arrow polynomials in Virtual KnotFolio [Mil20], which is a tool that can help identify prime virtual knots with respect to the [Gre04] table. The first and second cabled arrow polynomials are able to differentiate 2547 of the 2565 virtual knots with up to five crossings. Virtual KnotFolio was used by [GH] in their study of virtual knot mosaics.

Proposed research

- If all whiskers are of the same color, then this appears to be nearly identical to the $\mathfrak{su}(2)$ skein theory [CMW], which was used to define a version of Khovanov homology with "disorientations." Khovanov homology with disorientations might extend to a categorification of the arrow polynomial, and it might coincide with the categorifications in [DKM11].

  In another direction, at each primitive $2p$th root of unity, the trace of a link in $\Sigma \times I$ as an endomorphism in the Witten–Reshetikhin–Turaev $SU(2)$ TQFT gives a numerical invariant, and [MS] showed that a normalization of this invariant converges to a Laurent polynomial evaluated at the root of unity as $p \to \infty$. The arrow polynomial variables might arise from considering where the link sits in the skein module in families of stabilizations.

- The homological arrow polynomial is an "abelian" invariant from the skein module. More precise ones might arise from careful study of the character variety.

2.2 The Yamada polynomial

The Yamada polynomial is an invariant of ribbon graphs in S^3. It is generalized to virtual spatial graphs in [FM07] using the flow polynomial, and McPhail-Snyder and I generalized it to virtual spatial graphs in [MSM20] using our S-polynomial. At the same time, [DJK] found a two-variable Yamada polynomial for virtual spatial graphs by directly solving for skein relation coefficients.

Applying insights from the TQFT approach for the Bollobás–Riordan polynomial, I identified the two-variable Yamada polynomial as coming from the Frobenius algebra $\mathrm{Mat}_n(k) \otimes k^m$ interpolated at $m = n^{-2}(A + 2 + A^{-1})$. This has an immediate corollary that if the two-variable polynomial for a virtual spatial graph does not lie in $\mathbb{Z}[A^{\pm 1}]$, then it is not virtually equivalent to a classical spatial graph.

Proposed research

- The Yamada polynomial's Frobenius algebra is from $\mathrm{Rep}(\mathcal{U}_q(\mathfrak{sl}(2)))$. Quantum groups have resisted previous attempts to generalize to "virtual" quantum groups for virtual knot invariants. Finding the correct Frobenius algebra over the correct ring to explain the two-variable Bollobás–Riordan Yamada polynomial could illuminate a way forward.

- There are other "R-matrices" for this Frobenius algebra that give virtual spatial graph invariants, and it would be worth investigating whether they contain new information about the virtual spatial graph. Similarly, other symmetric Frobenius algebras may yield more virtual spatial graph invariants.

3 Other projects

3.1 Enumeration of knotted surfaces

Knotted (unoriented) smoothly embedded surfaces in $\mathbb{R}^4$ can be put into a form such that, with respect to the w coordinate, the minima occur at $w = -1$, the saddles at $w = 0$, and the maxima at $w = 1$. Furthermore, the cross section at $w = 0$ may be assumed to be in general position with respect to the projection onto $z = 0$. Hence, knotted surfaces can be given as a link diagram with special rigid degree-4 *hyperbolic* crossings such that both resolutions of the diagram are unlinks. The total number of crossings and degree-4 vertices is called the ch-index, and this structure was exploited by Yoshikawa to produce an enumeration of all prime knotted surfaces with diagrams having ch-index at most 10 [Yos94].

In joint work with Maggie Miller and Clayton McDonald, we are extending Yoshikawa's table to ch-index 11 with computational aid. The techniques we are attempting are general enough that we should be able to eventually continue to ch-index 12, though with considerably more effort.

3.2 Formalization of 3-manifold topology in a proof assistant

Lean is a system for computer-verified proof. There is a lively community of mathematicians who are developing a library for Lean called `mathlib` [The], and they have made significant progress toward the formalization of a complete undergraduate mathematics curriculum while also formalizing objects such as perfectoid spaces [BCM]. There is currently a project to formalize Smale's proof of the existence of sphere eversions.

I have been contributing to `mathlib` with an aim to eventually be able to formalize combinatorial 3-manifold topology. One piece of this has been the formalization of simple graphs, which will eventually lead toward multigraphs, surfaces, and eventually triangulated 3-manifolds. With a more immediate goal to show there are at least three distinct knot types, I recently also contributed the definition of racks and quandles along with Joyce's AdConj universal construction [Joy82]. Eventually, I plan to work on the loop and sphere theorems and normal surface theory.

References

[Abr97] Lowell Edward Abrams, *Frobenius algebra structures in topological quantum field theory and quantum cohomology*, ProQuest LLC, Ann Arbor, MI, 1997, Thesis (Ph.D.)–The Johns Hopkins University. MR 2695771

[BCM] Kevin Buzzard, Johan Commelin, and Patrick Massot, *Formalising perfectoid spaces*, arXiv:1910.12320v2 [cs.LO].

[Bir13] George D. Birkhoff, *A determinant formula for the number of ways of coloring a map*, Ann. of Math. (2) **14** (1912/13), no. 1-4, 42–46. MR 1502436

[BK] Hans U. Boden and Homayun Karimi, *The Jones-Krushkal polynomial and minimal diagrams of surface links*, arXiv:1908.06453v1 [math.GT].

[BN97] Dror Bar-Natan, *Lie algebras and the four color theorem*, Combinatorica **17** (1997), no. 1, 43–52, arXiv:q-alg/9606016v1 [q-alg]. MR 1466574

[BR01] Béla Bollobás and Oliver Riordan, *A polynomial invariant of graphs on orientable surfaces*, Proc. London Math. Soc. (3) **83** (2001), no. 3, 513–531. MR 1851080

[BR02] ———, *A polynomial of graphs on surfaces*, Mathematische Annalen **323** (2002), no. 1, 81–96. MR 1906909

[CK14] Abhijit Champanerkar and Ilya Kofman, *A survey on the Turaev genus of knots*, Acta Math. Vietnam. **39** (2014), no. 4, 497–514, arXiv:1406.1945v1 [math.GT]. MR 3292579

[CMW] David Clark, Scott Morrison, and Kevin Walker, *Fixing the functoriality of khovanov homology*, arXiv:math/0701339v2 [math.GT].

[Del07] P. Deligne, *La catégorie des représentations du groupe symétrique S_t, lorsque t n'est pas un entier naturel*, Algebraic groups and homogeneous spaces, Tata Inst. Fund. Res. Stud. Math., vol. 19, Tata Inst. Fund. Res., Mumbai, 2007, pp. 209–273. MR 2348906

[DFK+08] Oliver T. Dasbach, David Futer, Efstratia Kalfagianni, Xiao-Song Lin, and Neal W. Stoltzfus, *The Jones polynomial and graphs on surfaces*, Journal of Combinatorial Theory. Series B **98** (2008), no. 2, 384–399. MR 2389605

[DJK] Qingying Deng, Xian'an Jin, and Louis H. Kauffman, *The generalized yamada polynomials of virtual spatial graphs*, arXiv:1806.06462v3 [math.GT].

[DK09] H. A. Dye and Louis H. Kauffman, *Virtual crossing number and the arrow polynomial*, J. Knot Theory Ramifications **18** (2009), no. 10, 1335–1357, arXiv:http://arxiv.org/abs/0810.3858v3 [math.GT]. MR 2583800

[DKM11] Heather Ann Dye, Louis Hirsch Kauffman, and Vassily Olegovich Manturov, *On two categorifications of the arrow polynomial for virtual knots*, The mathematics of knots, Contrib. Math. Comput. Sci., vol. 1, Springer, Heidelberg, 2011, arXiv:http://arxiv.org/abs/0906.3408v3 [math.GT], pp. 95–124. MR 2777848

[DKR] Alexei Davydov, Liang Kong, and Ingo Runkel, *Field theories with defects and the centre functor*, arXiv:1107.0495v1 [math.QA].

[DO14] Zajj Daugherty and Rosa Orellana, *The quasi-partition algebra*, J. Algebra **404** (2014), 124–151, arXiv:1212.2596v1 [math.RT]. MR 3177889

[EMM13] Joanna A. Ellis-Monaghan and Iain Moffatt, *A Penrose polynomial for embedded graphs*, European Journal of Combinatorics **34** (2013), no. 2, 424–445. MR 2994409

[FHK] M. Fukuma, S. Hosono, and H. Kawai, *Lattice topological field theory in two dimensions*, arXiv:hep-th/9212154v1 [hep-th].

[FK09] Paul Fendley and Vyacheslav Krushkal, *Tutte chromatic identities from the Temperley-Lieb algebra*, Geom. Topol. **13** (2009), 709–741, arXiv:0711.0016v3 [math.CO]. MR 2469528

[FK10] ———, *Link invariants, the chromatic polynomial and the potts model*, Advances in Theoretical and Mathematical Physics **14** (2010), 507–540, arXiv:0806.3484v3 [math.CO].

[FM07] Thomas Fleming and Blake Mellor, *Virtual spatial graphs*, Kobe J. Math. **24** (2007), no. 2, 67–85, arXiv:math/0510158v2 [math.GT]. MR 2488753

[GH] Sandy Ganzell and Allison Henrich, *Virtual mosaic knot theory*, arXiv:2004.04790v1 [math.GT].

[GKRV16] Andrew Goodall, Thomas Krajewski, Guus Regts, and Lluis Vena, *A Tutte polynomial for maps*, arXiv:1610.04486v2 [math.CO].

[Gre04] Jeremy Green, *A table of virtual knots*, 2004, `https://www.math.toronto.edu/drorbn/Students/GreenJ/`.

[Hal40] Marshall Hall, *The position of the radical in an algebra*, Trans. Amer. Math. Soc. **48** (1940), 391–404. MR 2855

[HGR05] Laure Helme-Guizon and Yongwu Rong, *A categorification for the chromatic polynomial*, Algebr. Geom. Topol. **5** (2005), 1365–1388, arXiv:math/0412264v2 [math.CO]. MR 2171813

[HR05] Tom Halverson and Arun Ram, *Partition algebras*, European J. Combin. **26** (2005), no. 6, 869–921, arXiv:math/0401314v2 [math.RT]. MR 2143201

[Jan59] J. P. Jans, *On Frobenius algebras*, Ann. of Math. (2) **69** (1959), 392–407. MR 104711

[JHR06] Edna F Jasso-Hernandez and Yongwu Rong, *A categorification for the Tutte polynomial*, Algebr. Geom. Topol. **6** (2006), 2031–2049, arXiv:math/0512613v2 [math.CO]. MR 2263058

[Jon94] V. F. R. Jones, *The Potts model and the symmetric group*, Subfactors (Kyuzeso, 1993), World Sci. Publ., River Edge, NJ, 1994, pp. 259–267. MR 1317365

[Joy82] David Joyce, *A classifying invariant of knots, the knot quandle*, Journal of Pure and Applied Algebra **23** (1982), no. 1, 37–65.

[Kam] Naoko Kamada, *Span of the Jones polynomial of an alternating virtual link*, arXiv:math/0412074v1 [math.GT].

[Kau99] Louis H. Kauffman, *Virtual knot theory*, European Journal of Combinatorics **20** (1999), no. 7, 663–690. MR 1721925

[Kho] Mikhail Khovanov, *A categorification of the jones polynomial*, arXiv:math/9908171v2 [math.QA].

[Kru11] Vyacheslav Krushkal, *Graphs, links, and duality on surfaces*, Combin. Probab. Comput. **20** (2011), 267–287, arXiv:0903.5312v3 [math.CO]. MR 2769192

[KS93] Louis Kauffmann and H. Saleur, *An algebraic approach to the planar coloring problem*, Comm. Math. Phys. **152** (1993), no. 3, 565–590. MR 1213302

[Kup03] Greg Kuperberg, *What is a virtual link?*, Algebraic & Geometric Topology **3** (2003), 587–591. MR 1997331

[LP] Aaron D. Lauda and Hendryk Pfeiffer, *State sum construction of two-dimensional open-closed topological quantum field theories*, arXiv:math/0602047v2 [math.QA].

[LP08] ———, *Open-closed strings: two-dimensional extended TQFTs and Frobenius algebras*, Topology Appl. **155** (2008), no. 7, 623–666, arXiv:math/0510664v3 [math.AT]. MR 2395583

[Mar90] Paul P. Martin, *Representations of graph Temperley-Lieb algebras*, Kyoto University. Research Institute for Mathematical Sciences. Publications **26** (1990), no. 3, 485–503. MR 1068862

[Mar94] Paul Martin, *Temperley-Lieb algebras for nonplanar statistical mechanics—the partition algebra construction*, Journal of Knot Theory and its Ramifications **3** (1994), no. 1, 51–82. MR 1265453

[Mar96] ———, *The structure of the partition algebras*, Journal of Algebra **183** (1996), no. 2, 319–358. MR 1399030

[Mil20] Kyle Miller, *Virtual KnotFolio*, 2020, `https://kmill.github.io/knotfolio/`.

[MPS17] Scott Morrison, Emily Peters, and Noah Snyder, *Categories generated by a trivalent vertex*, Selecta Math. (N.S.) **23** (2017), no. 2, 817–868, arXiv:1501.06869v3 [math.QA]. MR 3624901

[MS] Julien Marché and Ramanujan Santharoubane, *Asymptotics of quantum representations of surface groups*, arXiv:1607.00664v1 [math.GT].

[MSM] Calvin McPhail-Snyder and Kyle Miller, *Planalg, a mathematica package for diagram categories*, `https://github.com/kmill/planalg`.

[MSM20] Calvin McPhail-Snyder and Kyle A. Miller, *Planar diagrams for local invariants of graphs in surfaces*, J. Knot Theory Ramifications **29** (2020), no. 1, 1950093, 49, arXiv:1805.00575v2 [math.GT]. MR 4079619

[Pen71] Roger Penrose, *Applications of negative dimensional tensors*, Combinatorial Mathematics and its Applications (Proc. Conf., Oxford, 1969), Academic Press, London, 1971, pp. 221–244. MR 0281657

[Sok05] Alan D. Sokal, *The multivariate Tutte polynomial (alias Potts model) for graphs and matroids*, Surveys in combinatorics 2005, London Math. Soc. Lecture Note Ser., vol. 327, Cambridge Univ. Press, Cambridge, 2005, arXiv:math/0503607v1 [math.CO], pp. 173–226. MR 2187739

[SP] Christopher J. Schommer-Pries, *The classification of two-dimensional extended topological field theories*, arXiv:http://arxiv.org/abs/1112.1000v2 [math.AT].

[Sto08] Marko Stošić, *Categorification of the dichromatic polynomial for graphs*, J. Knot Theory Ramifications **17** (2008), no. 1, 31–45, arXiv:math/0504239v2 [math.GT]. MR 2395978

[The] The mathlib Community, *The Lean mathematical library*, arXiv:1910.09336v2 [cs.LO].

[Thi87] Morwen B. Thistlethwaite, *A spanning tree expansion of the Jones polynomial*, Topology. An International Journal of Mathematics **26** (1987), no. 3, 297–309. MR 899051

[Tut47] W. T. Tutte, *A ring in graph theory*, Mathematical Proceedings of the Cambridge Philosophical Society **43** (1947), no. 1, 26–40.

[Tut54] ———, *A contribution to the theory of chromatic polynomials*, Canadian Journal of Mathematics **6** (1954), 80–91.

[Whi32] Hassler Whitney, *The coloring of graphs*, Ann. of Math. (2) **33** (1932), no. 4, 688–718. MR 1503085

[Wu82] F. Y. Wu, *The Potts model*, Rev. Modern Phys. **54** (1982), no. 1, 235–268. MR 641370

[Yam89] Shuji Yamada, *An invariant of spatial graphs*, Journal of Graph Theory **13** (1989), no. 5, 537–551. MR 1016274

[Yos94] Katsuyuki Yoshikawa, *An enumeration of surfaces in four-space*, Osaka J. Math. **31** (1994), no. 3, 497–522. MR 1309400