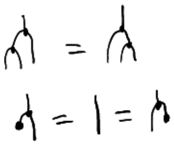
Bialgebra B over k

Associative unital algebra

 : $B \otimes B \to B$

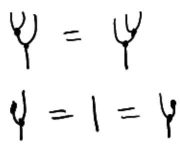
 : $k \to B$



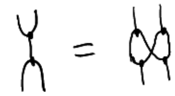
Coassociative counital coalgebra

 : $B \to B \otimes B$

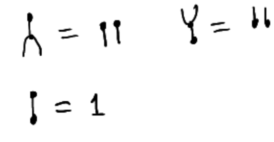
 : $B \to k$



Compatibility:

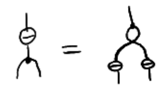


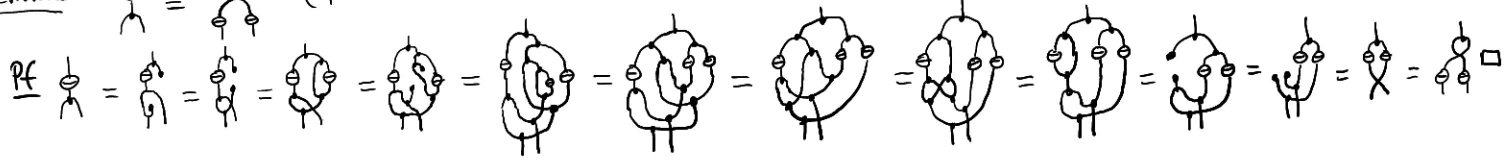
(comultiplication is a homomorphism, or multiplication is a cohomomorphism)

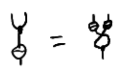


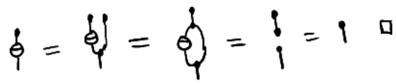
Hopf algebra

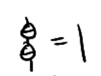
H is a bialgebra with antipode  : $H \to H$ satisfying 

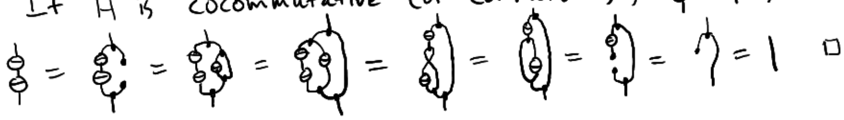
Lemma  ( is an antihomomorphism)

Pf  □

Corollary  . **Pf** Flip the diagrams over.

Lemma  **Pf**  □

Lemma If H is cocommutative (or commutative), , thus  is an isomorphism

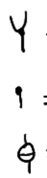
Pf  □

Group algebra

Let G be a group, k a field. $H = k[G]$ is a Hopf algebra.

 is multiplication.

 $= 1$



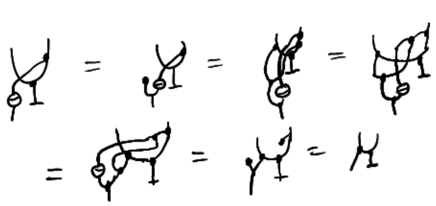
$= g \mapsto g \otimes g$

$= g \mapsto 1$

$= g \mapsto g^{-1}$

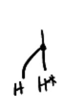
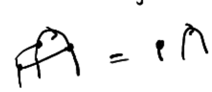
} defined on basis $g \in G$

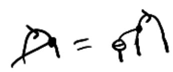
Let $\perp \in H$ be $\frac{1}{|G|}\sum_{g\in G} g$.  $= h \mapsto \frac{1}{|G|}\sum_{g\in G} g = h \mapsto \frac{1}{|G|}\sum_{g\in G} hg =$  so $\perp \in \mathrm{Hom}_H(k,H)$



Dual:  : $H \otimes H^* \to k$

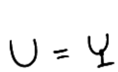
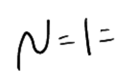
$= g \otimes \varphi \mapsto \langle \varphi, g \rangle$

 defined by 

Forces 

Then:

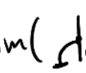
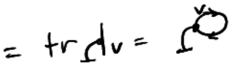
so  $\in \mathrm{Hom}_H(H^*, H)$.

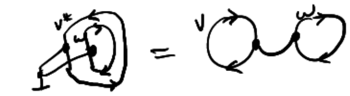
And, is isomorphism if G finite. Hence, we let  and  .  and $O = |G|$

$\cup = \frac{1}{|G|}\sum_{g\in G} g\,\phi = \frac{1}{|G|}\sum_{g\in G} g = \perp$  $= \frac{1}{|G|}\sum_{g\in G} g^{-1} = \perp$

Let V be a representation of H.

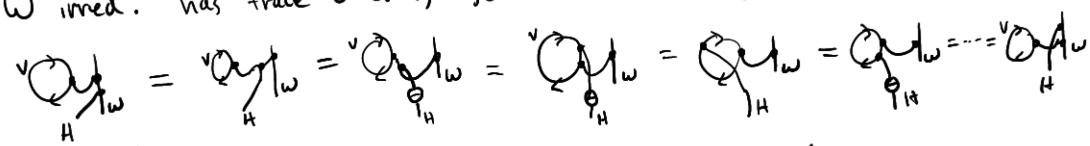
$\chi_V =$  $: H \to k$ is a character.

$V^G = \{ v \in V \mid gv = v \ \forall g \in G \}$. $V^G = \left(\frac{1}{|G|}\sum_{g\in G} g\right) V$ so im(). $\dim V^G = $ tr 

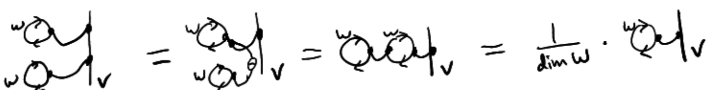
$\dim \mathrm{Hom}_G(V, W) = \dim (V^* \otimes W)^G =$ 

If V, W irred., $= \begin{cases} 1 & \text{if } V \cong W \\ 0 & \text{if } V \not\cong W \end{cases}$ (Schur's lemma for k algebraically closed)

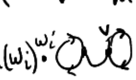
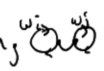
 $: W \to W$ for W irred. has trace 0 or 1, so $\dim(W) \cdot$  is id_W

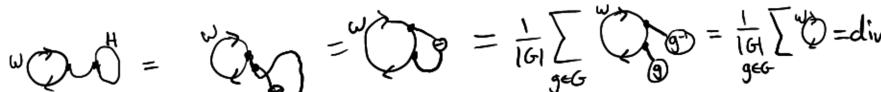
Check: endomorphism. 

Prop Let W be irreducible. $\dim(W) \cdot$  is projector onto W isotypic component of V.

Pf , thus multiplying by $\dim(W)$ gives projector.

If $V = m_1 W_1 \oplus \cdots \oplus m_n W_n$ is decomposition into distinct irreducible factors,

$\dim(W_i)$  $= \dim(W_i) \cdot \sum_{j=1}^{n} m_j$  $= \dim(W_i) m_i$, hence image of projector is all of $m_i W_i$. □

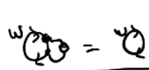
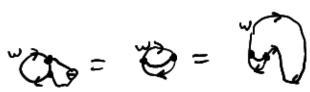
Suppose W is irreducible. Then  $= \dim W$.

Thus, if $H = m_1 W_1 \oplus m_2 W_2 \oplus \cdots \oplus m_n W_n$ is a decomposition into distinct irreducible representations,

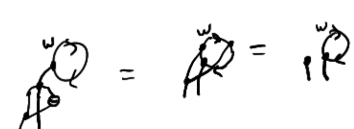
1) This is a complete set
2) $m_i = \dim W_i$

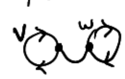
Corollary $|G| = \sum_{i=1}^{n} (\dim W_i)^2$.

The projectors imply H has such a decomposition. Finite-dimensional $\Rightarrow$ every subrepresentation of H has an irreducible subrepresentation, and if π is a projector, $\mathrm{id} - \pi$ projects to the complement and is also a homomorphism.

Cor to  : , so the map  is id_W.

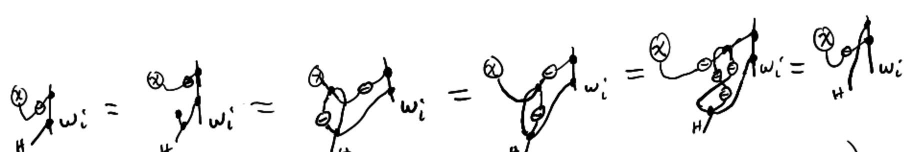
Characters again:

Characters are functions of the conjugacy class: 

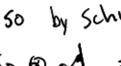
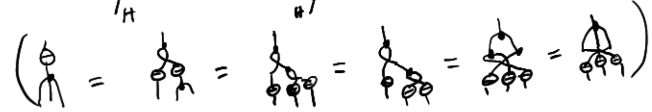
Ireds are orthonormal by the symmetric form $\langle \chi_V, \chi_W \rangle =$  $= \begin{cases} 1 & \text{if } V \cong W \\ 0 & \text{if } V \not\cong W \end{cases}$

Suppose  is a function constant on conjugacy classes that is orthogonal to all irreducible W_i.

That is,  $= 0$ for all i.

(1)  is an endomorphism: 

so by Schur's lemma $= 0$

(2) So  $= 0$, but $= \frac{1}{|G|} \sum_{g \in G} \chi(g) g^{-1}$ and g's are linearly indep., so $\chi = 0$. ()

Therefore $\chi_{W_1}, \ldots, \chi_{W_n}$ are a basis of characters, and $n = \#$ conj. classes of G.

"Orthogonality of columns"

For $g \in G$, let C_g = conjugacy class of g and $\eta_g(h) = \begin{cases} 1 & \text{if } h \in C_g \\ 0 & \text{otherwise} \end{cases}$. η_g is a function on conj. classes

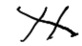
so $\eta_g = \sum_i a_i \chi_i$ for some constants a_i. $\langle \chi_i, \eta_g \rangle =$  $= \sum_j a_j$  $= a_i$,

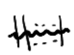
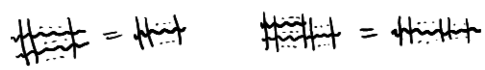
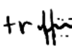
and  $= \frac{1}{|G|} \sum_{h \in G} \chi_i(h^{-1}) \eta_g(h) = \frac{|C_g|}{|G|} \chi_i(g^{-1})$, so $\eta_g = \frac{1}{|G|} \sum_i |C_g| \chi_i(g^{-1}) \chi_i$

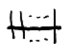
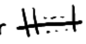
If g, h conjugate, get $1 = \frac{1}{|G|} \sum_i |C_g| \chi_i(g^{-1}) \chi_i(h)$. Else, get $0 = \frac{1}{|G|} \sum_i |C_g| \chi_i(g^{-1}) \chi_i(h)$.

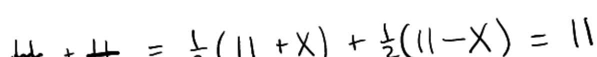
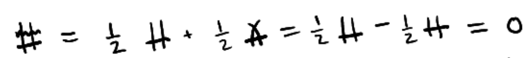
Thus, $\sum_i \chi_i(g^{-1}) \chi_i(h) = \begin{cases} |G|/|C_g| & \text{if } h \in C_g \\ 0 & \text{otherwise} \end{cases}$.

Tensor powers

S_n acts on n parallel strands, giving an endomorphism. ex: $(1\,2\,3) \mapsto$ 

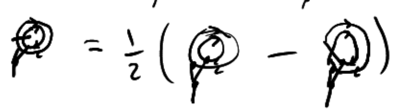
 $= \frac{1}{n!} \sum_{\sigma \in S_n} \sigma_*$. im  $\cong \mathrm{Sym}^n V$ , tr  $= \dim \mathrm{Sym}^n V = \binom{\dim V + n - 1}{n}$

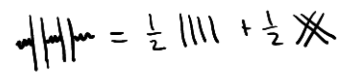
 $= \frac{1}{n!} \sum_{\sigma \in S_n} (-1)^\sigma \sigma_*$. im  $\cong \Lambda^n V$ Similar identities, tr  $= \binom{\dim V}{n}$

ex  and 

so these are a pair of projectors decomposing the tensor square ($V^{\otimes 2} \cong \mathrm{Sym}^2 V \oplus \Lambda^2 V$)

Characters: $\chi_{sym} =$ , so $\chi_{sym}(g) = \frac{1}{2}(\chi(g)^2 + \chi(g^2))$

$\chi_{alt} =$  so $\chi_{alt}(g) = \frac{1}{2}(\chi(g)^2 - \chi(g^2))$.

To represent powers of powers, ex $\mathrm{Sym}^2 V^{\otimes 2}$ by 

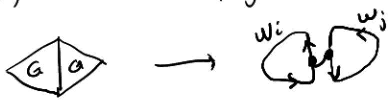
Graph invariant

(I think this is what Kevin was talking about Mar 21, 2017.)

Suppose a graph is given oriented faces (not necessarily so that the result is a mfld)

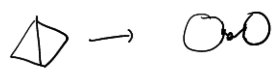
Put a loop in each face colored by a representation, ex 

At edges, use trivial projector:  , etc.

ex 

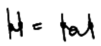
$= \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i\neq j \end{cases}$

ex Colored by H,

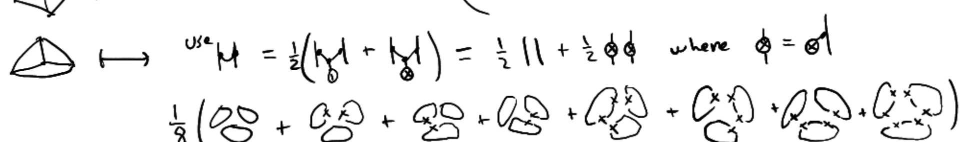
 $= \sum_i (\dim w_i)^2$

ex $H = \mathbb{C}[C_2]$ so

	1	x
χ_1	1	1
χ_2	1	-1

For C_2, 

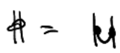
 $\mapsto 1^2+1^2=2$ or $\frac{1}{2}(\ldots + \ldots) = \frac{1}{2}(4+0)=2$



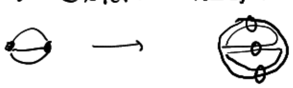
$\frac{1}{8}(\ldots)$

$= \frac{1}{8}(8+0+0+0+0+0+0+8)$

$= 2$

Let  .

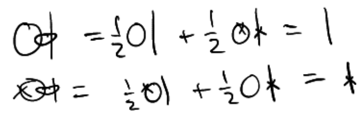
If use exterior face, too,



$= \frac{1}{2}$  $+ \frac{1}{2}$ 

$= \frac{1}{2}$  $+ \frac{1}{2}$ 

$= 1 + \frac{1}{2}$  $= 2$

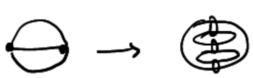


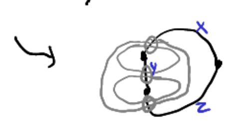
Same as counting solutions to $\begin{cases} x+y=0 \\ x+z=0 \\ y+z=0 \end{cases}$ mod 2

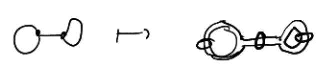
$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

Rank 2 → |Null| = 2

Using C_n instead, it is almost the flow polynomial of the dual graph. (Does not care about vanishing)

 says $\begin{cases} x-y=0 \\ x-z=0 \\ y-z=0 \end{cases}$ mod n. So $x=y=z$ mod n ⇒ $= n$.





 self-dual. n^3 from homology. $2 = v-e+f$. dual has f verts, e edges

$1-b_1 = f-e = (2-v+e)-e = 2-v$

$b_1 = v-1$

So just n^{v-1}.